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Zolotarev ω-Polynomials in $W^r H^\omega[0, 1]$

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The main result of this paper characterizes generalizations of Zolotarev polynomials as extremal functions in the Kolmogorov–Landau problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^r H^\omega[0, 1], \; \|f\|_{\mathbb{C}[0,1]} \leqslant B, \tag{$\star$}$$

where $\omega(t)$ is a concave modulus of continuity, $r, m\colon 1 \leqslant m \leqslant r$, are integers, and $B \geqslant B_0(r, m, \omega)$.

We show that the extremal functions Z_B have $r+1$ points of alternance and the full modulus of continuity of $Z_B^{(r)}$: $\omega(Z_B^{(r)}; t) = \omega(t)$ for all $t \in [0, 1]$. This generalizes the Karlin's result on the extremality of classical Zolotarev polynomials in the problem ($\star$) for $\omega(t) = t$ and all $B \geqslant B_r$. © 1997 Academic Press

0. INTRODUCTION

0.1. *Classical Zolotarev Polynomials*

The family $\{Z_B\}_{B>0}$ of classical Zolotarev polynomials of degree $r+1$ on the interval $[0, 1]$ can be characterized as follows:

for any $B > 0$, there exist points $\{\tau_i(B)\}_{i=0}^r$, $0 =: \tau_0(B) < \cdots < \tau_r(B) \leqslant 1$ and such a polynomial $Z_B(x) = x^{r+1}/(r+1)! + \sum_{i=0}^r a_i x^i$ that

$$Z_B(\tau_i(B)) = (-1)^{r+1+i} \|Z_B\|_{\mathbb{C}[0, \tau_r(B)]} = (-1)^{r+1+i} B, \qquad i = 0, ..., r. \tag{0.1}$$

Let $C_r(x)$ be the Chebyshev polynomial of degree $r+1$ with the leading coefficient $1/(r+1)!$:

$$C_r(x) = \frac{2^{-2r-1}}{(r+1)!} \cos[(r+1) \arc\cos(2x-1)], \qquad x \in [0, 1]. \tag{0.2}$$

Let $L_r := \|C_r\|_{\mathbb{C}[0,1]} = 2^{-2r-1}/(r+1)!$ By (0.2),

$$\{T_i = \tfrac{1}{2}(1 + \cos(\pi i/(r+1)))\}_{i=0}^{r+1}$$

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is the collection of alternance points of $C_r(x)$ on the interval $[0,1]$:

$$C_r(T_i)=(-1)^{r+1+i}\|C_r\|_{\mathbb{C}[0,1]}=(-1)^{r+1+i}L_r, \qquad i=0, ..., r+1. \quad (0.3)$$

Let $K_r:=L_r\cdot T_r^{-(r+1)}$. For $0<B\leqslant K_r$, the Zolotarev polynomial $Z_B(x)$ is a properly rescaled and dylated Chebyshev polynomial

$$Z_B(x)=\lambda_B^{-(r+1)}C_r(\lambda_B x), \qquad \lambda_B:=\frac{L_r}{B}, \quad (0.4)$$

with $r+1$ points of alternance $\{\tau_i(B)=\lambda_B T_i\}_{i=0}^r$ on the interval $[0,\tau_r(B)]$. In the case $B\in[L_r,K_r]$, the collection $\{\tau_i(B)\}_{i=0}^r$ is the set of alternance points of the function Z_B on the entire interval $[0,1]$:

$$Z_B(\tau_i(B))=(-1)^{r+1+i}\|Z_B\|_{\mathbb{C}[0,1]}=(-1)^{r+1+i}B, \qquad i=0, ..., r. \quad (0.5)$$

For $B>K_r$, the Zolotarev polynomial $Z_B(t)$ admits an expression in terms of elliptic functions [1, 12].

For $n\in\mathbb{N}$, let us introduce the Sobolev class

$$W_\infty^n[a,b]=\{f\in\mathbb{C}^{n-1}[a,b]\,|\,f^{(n-1)} \text{ is abs. cont. and } \|f^{(n)}\|_{\mathbb{L}_\infty[a,b]}\leqslant 1\}. \quad (0.6)$$

S. Karlin [7, p. 419] showed that the Zolotarev polynomial Z_B enjoys the extremal property

$$(-1)^{r+1+m}Z_B^{(m)}(0)=\sup\{f^{(m)}(0)\,|\,f\in W_\infty^{r+1}[0,1],\ \|f\|_{\mathbb{C}[0,\tau_r(B)]}\leqslant B\}. \quad (0.7)$$

In view of properties (0.5) and (0.7), in the case $B\in[L_r,K_r]$, the function Z_B is extremal in the Kolmogorov–Landau problem

$$f^{(m)}(0)\to\sup, \qquad f\in W_\infty^{r+1}[0,1], \quad \|f\|_{\mathbb{C}[0,1]}\leqslant B. \quad (0.8)$$

Definition 0.1. Let f be a continuous function on the interval $[a,b]$. The function

$$\omega(f;t)=\sup_{\substack{x,\,y\in[a,b]\\ |x-y|\leqslant t}}|f(x)-f(y)|, \qquad t\in[0,b-a], \quad (0.9)$$

is called *the modulus of continuity of the function f.*

The functional class $W_\infty^{r+1}[0,1]$ is defined by the constraint $\|f^{(r+1)}\|_{\mathbb{L}_\infty(I)}\leqslant 1$, equivalent to inequalities $\omega(f^{(r)},t)\leqslant t$ for all $t\in[0,1]$. In our generalizations, we consider the classes of functions defined by the continuum of inequalities of the form $\omega(f^{(r)}, t) \leqslant \omega(t)$, for $t \in [0, 1]$ and some fixed *concave modulus of continuity* ω. Such constraints enable us not only to control upper bounds of the function $f^{(r+1)}$ but also to retain information on the order of growth of the rth derivative $f^{(r)}$.

This discussion leads us to the definition of functional classes $W^r H^\omega[0, 1]$ with a majorizing modulus of continuity and generalizations of Zolotarev polynomials in $W^r H^\omega[0, 1]$.

0.2. *Functional Classes* $W^r H^\omega[a, b]$

Let us introduce the notion of a concave modulus of continuity on the half-line $\mathbb{R}_+$.

Definition 0.2. A function $\omega(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$, is called *a concave modulus of continuity on* $\mathbb{R}_+$, if the following conditions are satisfied:

$$\begin{aligned} &(1) \quad \omega(0) = 0; \\ &(2) \quad \omega(t_1) \leqslant \omega(t_2), \qquad \text{if} \quad 0 \leqslant t_1 \leqslant t_2; \\ &(3) \quad \omega(\alpha t_1 + (1-\alpha)\, t_2) \geqslant \alpha\omega(t_1) + (1-\alpha)\, \omega(t_2), \qquad \text{for all} \quad \alpha \in [0, 1], \\ &\qquad \text{and} \qquad t_1, t_2 \in \mathbb{R}_+. \end{aligned} \tag{0.10}$$

Definition 0.3. Let $\omega(t)$ be a concave modulus of continuity on $\mathbb{R}_+$. *The functional class* $W^r H^\omega[a, b]$ is defined as

$$W^r H^\omega[a, b] := \{x \in \mathbb{C}^r[a, b] \mid \omega(x^{(r)}; t) \leqslant \omega(t),\ t \in [0, b-a]\}. \tag{0.11}$$

In the case $r = 0$ we also use the notations

$$H^\omega[a, b] := W^0 H^\omega[a, b], \qquad H^\omega_0[a, b] := \{f \in H^\omega[a, b] \mid f(a) = 0\}. \tag{0.12}$$

The standard Sobolev class $W^{r+1}_\infty[a, b]$ is a particular case of the class $W^r H^{\tilde\omega}[a, b]$ with $\tilde\omega(t) = t$. Another example is provided by *the Hölder modulii of continuity* $\omega_\alpha(t) = t^\alpha$, $0 < \alpha \leqslant 1$. In this case, we denote

$$W^r H^\alpha[a, b] := W^r H^{\omega_\alpha}[a, b]. \tag{0.13}$$

We mention that classes $W^r H^\omega[a, b]$ were introduced in 1946 by S. M. Nikol'skii [10] in connection with approximation of functions by Fourier sums.

0.3. *Zolotarev ω-Polynomials*

Fix $r, m \in \mathbb{N}: 1 \leqslant m \leqslant r$ and a concave modulus of continuity ω on $\mathbb{R}_+$. Our main goal in this paper is to construct a family $\{Z_B = Z_{B,r,m,\omega}\}_{B>0}$ of functions endowed with the properties

$$\begin{aligned}
&(1) \quad \text{there exist such points } \{\tau_i(B)\}_{i=0}^r,\ 0=\tau_0(B)< \cdots <\tau_r(B)\leqslant 1, \text{ that} \\
&\qquad Z_B(\tau_i(B)) = (-1)^{i+m}\|Z_B\|_{\mathbb{C}[0,\tau_r(B)]}, \qquad i=0,...,r; \\
&(2) \quad \omega(Z_B^{(r)}; t) = \omega(t), \qquad 0 \leqslant t \leqslant 1; \\
&(3) \quad Z_B^{(m)}(0) = \sup\{f^{(m)}(0) \mid f \in W^rH^\omega[0,1],\ \|f\|_{\mathbb{C}[0,\tau_r(B)]} \leqslant B\}.
\end{aligned} \tag{0.14}$$

If ω is a linear modulus of continuity $\omega_M(t) = Mt$, $M>0$, then

$$\omega(f^{(r)}; t) = \omega_M(t), \quad t\in[0,1] \Leftrightarrow f^{(r)}(t) = C \pm Mt, \quad C\in\mathbb{R}, \quad t\in[0,1],$$

i.e., f is a polynomial of degree $r+1$ with the leading coefficient $M/(r+1)!$ Therefore, it is natural that functions Z_B with features (0.14) generalizing the properties of classical Zolotarev polynomials will be called *the Zolotarev ω-polynomials.*

0.4. *Organization of the Paper*

In Section 1 we list auxiliary results used in our constructions: the Borsuk theorem, the Chebyshev theorem, the Korneichuk lemma with corollaries, and some other special technical propositions.

Section 2 contains the proof of the main result of this paper—Theorem 2.1 describing Zolotarev ω-polynomials $Z_B = Z_{B,\omega,r,m}$ of the norm B.

A number of corollaries from Theorem 2.1 are derived in Section 3. We show the existence of such a constant $M = M_{\omega,r,m}$ that for all $B>M$, the Zolotarev function Z_B is extremal in the Kolmogorov–Landau problem

$$f^{(m)}(0) \to \sup, \qquad f\in W^rH^\omega[0,1], \qquad \|f\|_{\mathbb{C}[0,1]} \leqslant B. \tag{0.15}$$

In the special case of the Hölder modulus of continuity $\omega_\alpha(t) = t^\alpha$, we construct the Chebyshev ω-polynomial $C(x) = C_{\alpha,r,m}(x)$ with a complete $(r+2)$-alternance and extremal in (0.15) for $B = L := \|C\|_{\mathbb{C}[0,1]}$. Then, we show the extremality of Z_B in the problem (0.15) for all $B \geqslant L$.

For all sufficiently large $B>0$, we demonstrate the continuous dependence of the alternance points $\{\tau_i(B)\}_{i=0}^r$ of Z_B on B and the uniqueness of solutions of the problem (0.15).

Finally, using specific features of the class $W^2H^\omega[0,1]$, we also describe the complete (for all $B>0$) set of extremal functions in the problem (0.15) for $r=2$.

1. AUXILIARY RESULTS

The Korneichuk lemma describes extremal functions in the problem

$$\int_a^b h(t)\,\psi(t)\,dt \to \sup, \qquad h \in H^\omega[a, b], \tag{1.1}$$

where ψ is the derivative of *a simple kernel* on $[a, b]$.

DEFINITION 1.1. Let the kernel $\psi(\cdot) \in \mathbb{L}_1[a, b]$ be endowed with the properties: for some $a', b' : a < a' \leqslant b' < b$,

$$\begin{aligned} &\text{(i)} && \psi(x) < 0, && \text{for a.e.} \quad x \in [a, a']; \\ &\text{(ii)} && \psi(x) = 0, && \text{for a.e.} \quad x \in [a', b']; \\ &\text{(iii)} && \psi(x) > 0, && \text{for a.e.} \quad x \in [b', b]; \\ &\text{(iv)} && \int_a^b \psi(x)\,dx = 0. \end{aligned} \tag{1.2}$$

Then the kernel $\Psi(x) = \chi \int_a^x \psi(t)\,dt$, $a \leqslant x \leqslant b$, $\chi \in \{-1, 1\}$, fixed, is called *a simple kernel.*

Notice that for any simple kernel Ψ, the equation $|\Psi(t)| = y$, for $0 < y < \|\Psi\|_{\mathbb{C}[a, b]}$, has precisely two solutions: $\alpha_y \in (a, a')$ and $\beta_y \in (b', b)$. The quantitative solution of the problem (1.1) will be given in terms of *the rearrangement of the simple kernel* Ψ.

DEFINITION 1.2. Let $\Psi(x)$, $a \leqslant x \leqslant b$, be a simple kernel. Let the function $r : [a, (a' + b')/2] \to [(a' + b')/2, b]$ be derived from equations

$$\begin{aligned} \Psi(t) &= \Psi(r(t)), && t \in [a, a'], \\ r(t) &= a' + b' - t, && t \in [a', (a' + b')/2]. \end{aligned} \tag{1.3}$$

Then, *the rearrangement* $\Re(\Psi; t)$, $0 \leqslant t \leqslant b - a$, *of the simple kernel* $\Psi(t)$ is defined as

$$\Re(\Psi; t) := \begin{cases} \|\Psi\|_{\mathbb{C}[a, b]}, & t \in [0, b' - a'], \\ |\Psi(y_t)|, & t \in (b' - a', b - a], \\ \qquad \text{where} \quad y_t \in [a, a'] & \text{is such that} \quad r(y_t) - y_t = t. \end{cases} \tag{1.4}$$

We also need the following properties of *concave modulii of continuity* ω [9, pp. 263, 264].

PROPOSITION 1.1. *Let ω be a concave modulus of continuity on $\mathbb{R}_+$. Then,*

(a) *at any point $x>0$, ω has one-sided derivatives*

$$\omega'_-(x)=\lim_{h\to 0+}\frac{\omega(x)-\omega(x-h)}{h}, \qquad \omega'_+(x)=\lim_{h\to 0+}\frac{\omega(x+h)-\omega(x)}{h};$$

(b) *each of the functions ω'_+ and ω'_- does not increase on $(0,+\infty)$, and*

$$\omega'_+(x)\leqslant\omega'_-(x), \qquad x>0;$$

(c) *ω is an absolutely continuous function on $\mathbb{R}_+$.*

In this paper we make the following choice from the equivalence class of summable functions defining the nonincreasing derivative ω' *everywhere* on $\mathbb{R}_+$.

DEFINITION 1.3. Let ω be a concave modulus of continuity on $\mathbb{R}_+$. We put

$$\omega'(u):=\tfrac{1}{2}[\omega'_+(u)+\omega'_-(u)], \qquad u>0. \tag{1.5}$$

The following result [9, pp. 302–307] describes the derivative of extremal functions of the problem (1.1).

LEMMA 1.2. *Let $\Psi(t):=\chi\int_a^t\psi(y)\,dy$, $a\leqslant t\leqslant b$, $\chi\in\{-1,1\}$, be a simple kernel whose derivative satisfies* (1.2). *Let the function $r\colon[a,c]\to[c,b]$, $c:=(a'+b')/2$, be defined by* (1.3), *and the rearrangement $\mathfrak{R}(\Psi;t)$ be introduced in* (1.4). *Let $\omega(t)$ be a concave modulus of continuity on $[0,b-a]$. Then,*

$$M_\omega(\psi):=\sup_{f\in H^\omega[a,b]}\int_a^b f(t)\,\psi(t)\,dt=\int_0^{b-a}\mathfrak{R}(\Psi;t)\,\omega'(t)\,dt, \tag{1.6}$$

and the upper bound in (1.6) *is attained on the functions whose derivative is given by the formula*

$$\frac{d}{dx}f_0(x)=\begin{cases}\omega'(r(x)-x), & a\leqslant x\leqslant c,\\ \omega'(x-r^{-1}(x)), & c\leqslant x\leqslant b.\end{cases} \tag{1.7}$$

Note that extremal functions of the problem (1.1) are determined up to a constant, since $\int_a^b\psi(t)\,dt=0$. Therefore,

$$\sup_{h\in H^\omega[a,b]}\int_a^b h(t)\,\psi(t)\,dt=\sup_{h\in H_0^\omega[a,b]}\int_a^b h(t)\,\psi(t)\,dt. \tag{1.8}$$

From the formula (1.7) it follows that if $a' = b' = c$, then the derivative of extremal function of the problem (1.1) is determined uniquely by (1.7).

We mention some corollaries from Lemma 1.2 used in this paper.

COROLLARY 1.2.1. *Let the function f_0 be defined by* (1.7). *Then, f_0 has the full modulus of continuity on the interval* $[0, b-a]$:

$$\omega(f_0; t) = \omega(t), \qquad 0 \leqslant t \leqslant b-a. \tag{1.9}$$

Proof. By (1.7), for any $x\colon 0 \leqslant x \leqslant c := \frac{1}{2}(a'+b')$, we have

$$\begin{aligned} f_0(r(x)) - f_0(x) &= \int_c^{r(x)} \omega'(u - r^{-1}(u))\, du - \int_c^x \omega'(r(u)-u)\, du \\ &= \int_c^x \omega'(r(u)-u)\, r'(u)\, du - \int_c^x \omega'(r(u)-u)\, du \\ &= \int_c^x \omega'(r(u)-u)\, d(r(u)-u) \\ &= \omega(r(x)-x). \end{aligned} \tag{1.10}$$

It remains to notice that the function $r(t)-t$ increases from 0 to $b-a$, as t decreases from c to 0. ∎

COROLLARY 1.2.2. *Let $0 < \alpha \leqslant 1$. Let $\Psi(t)$ be a simple kernel on $[0, b]$, and f be an extremal function in the problem*

$$\int_0^b h(t)\, \Psi'(t)\, dt \to \sup, \qquad h \in H^\alpha[0, b].$$

Then, for any $\sigma > 0$, the function $h_\sigma(t) = \sigma^\alpha f(t/\sigma)$ is extremal in the problem

$$\int_0^{\sigma b} h(t)\, \Psi'(t/\sigma)\, dt \to \sup, \qquad h \in H^\alpha[0, \sigma b].$$

The proof of Corollary 1.2.2 follows either from the form (1.7) of the derivative of extremal function in the problem (1.1) or from the observation

$$f(t) \in H^\alpha[0, b] \Leftrightarrow \sigma^\alpha f(t/\sigma) \in H^\alpha[0, \sigma b].$$

For the proof of the following limiting property of solutions of problems (1.1), the reader is referred to [2].

LEMMA 1.3. *Let $\mathbb{S}$ be a compact of $\mathbb{R}^d$ and the family of simple kernels Ψ_s, $s \in \mathbb{S}$ be endowed with the following properties.*

(i) *The endpoints* a_s *and* b_s *are continuous functions of* s *on* $\mathbb{S}$, *and*

$$a_s < a < b < b_s, \qquad s \in \mathbb{S}, \qquad \text{for some} \quad a < b.$$

(ii) *The zero-interval of the kernel* ψ_s *degenerates into a point, i.e.,* $a'_s = b'_s$, *for all* $s \in \mathbb{S}$.

(iii) *The family* $\{\psi_s(t) = \Psi'_s(t)\}_{s \in \mathbb{S}}$ *depends continuously on* s *on* $\mathbb{S}$ *in the integral metrics in the following sense: for all* $s \in \mathbb{S}$,

$$\|\bar{\psi}_{s'} - \bar{\psi}_s\|_{\mathbb{L}_1[a,b]} \to 0, \qquad \text{as} \quad s' \to s,$$

where

$$\bar{\psi}_s(x) := \psi_s\left(\frac{b_s - a_s}{b - a}(x - a) + a_s\right), \qquad a \leqslant x \leqslant b, \quad s \in \mathbb{S}.$$

Let x_s *be the solution of the problem*

$$\int_{a_s}^{b_s} f(t)\, \psi_s(t)\, dt \to \sup, \qquad f \in H^\omega[a_s, b_s], \quad f(a_s) = 0.$$

Then, functions x_s *depend continuously on* s *on* $\mathbb{S}$ *in the uniform metrics, i.e., for all* $s \in \mathbb{S}$,

$$\|x_{s'} - x_s\|_{\mathbb{C}[\max\{a_s, a_{s'}\}, \min\{b_s, b_{s'}\}]} \to 0, \qquad \text{as} \quad s' \to s.$$

The proof of Theorem 2.1 is based on the following topological result known as *the Borsuk Antipodality Theorem* (cf. [5], [6]).

THEOREM 1.4. *Let* $\mathbb{S}^n = \{\xi : \xi \in \mathbb{R}^{n+1} \mid \|\xi\| = r\}$, *where* $\|\cdot\|$ *is some norm in* $\mathbb{R}^{n+1}$, *and let* $\eta : \mathbb{S}^n \to \mathbb{R}^n$, $\eta(\xi) = \{\eta_1(\xi), \eta_2(\xi), ..., \eta_n(\xi)\}$, *be a continuous and odd* ($\eta(-\xi) = -\eta(\xi)$) *vector field on* $\mathbb{S}^n$. *Then, there exists a vector* $\bar{\xi} \in \mathbb{S}^n$ *such that* $\eta(\bar{\xi}) = 0$.

The polynomial of the best approximation for a given continuous function is characterized as follows [9, p. 48].

THEOREM 1.5. *Let* $f \in \mathbb{C}[a, b]$. *Then,*

(a) *there exists a unique polynomial* $p_f(t) = \sum_{i=0}^n a_i(f)\, t^i$ *of the best approximation for* f *on the interval* $[a, b]$ *among the polynomials of degree* n, *i.e.,*

$$\|f - p_f\|_{\mathbb{C}[a,b]} = \min_{p \in \mathscr{P}_n} \|f - p\|_{\mathbb{C}[a,b]},$$

where $\mathscr{P}_n$ *is the linear space of polynomials of degree* n;

(b) *the polynomial p_f is the polynomial of the best approximation for f among the polynomials of degree n, if and only if there exist $n+2$ points $\{x_k\}_{k=1}^{n+2}$, $a \leqslant x_1 < x_2 < \cdots < x_{n+2} \leqslant b$, such that*

$$(f-p_f)(x_i) = (-1)^i \, \xi \, \|f-p_f\|_{\mathbb{C}[a,b]}, \qquad i=1, ..., n+2, \tag{1.11}$$

where $\xi = \xi(f) \in \{-1, 1\}$, fixed.

PROPOSITION 1.6. *Let $f \in \mathbb{C}^r[a,b]$. If f has r zeroes (counting multiplicities), then*

$$\|f^{(k)}\|_{\mathbb{C}[a,b]} \leqslant [b-a]^{r-k} \, \|f^{(r)}\|_{\mathbb{C}[a,b]}, \qquad k=0, ..., r. \tag{1.12}$$

Proof. By Rolle's theorem, the derivative $f^{(k)}(t)$ has a zero ξ_k on $[a,b]$ for $k=0, ..., r-1$. Then, $f^{(k)}(x) = \int_{\xi_k}^{x} f^{(k+1)}(t)\, dt$, $a \leqslant x \leqslant b$, $k=0, ..., r-1$. Thus,

$$\|f^{(k)}\|_{\mathbb{C}[a,b]} \leqslant (b-a) \, \|f^{(k+1)}\|_{\mathbb{C}[a,b]}, \tag{1.13}$$

implying (1.12).∎

We also need the following result [9, p. 92] on the existence of a polynomial perfect spline satisfying the zero boundary conditions.

PROPOSITION 1.7. *Let $r \in \mathbb{N}$. There exists a unique perfect polynomial spline*

$$Y_r(x) = \frac{x^r}{r!} + \frac{2}{r!} \sum_{i=1}^{r} (-1)^i \, (x-t_i)_+^r + \sum_{i=0}^{r-1} a_i x^i$$

with r knots $\{t_i\}_{i=1}^r$, $0 < t_1 < \cdots < t_r < 1$, satisfying the boundary conditions $Y_r^{(k)}(0) = Y_r^{(k)}(1) = 0$, $k=0, ..., r-1$. In addition, $Y_r(x) > 0$, $x \in (0,1)$.

The following two results play an important role in the final phase of the proof of Theorem 2.1.

PROPOSITION 1.8. *Let $f \in W^r H^\omega[0,1]$. Then, there exist such constants $E_1 = E_1(r)$ and $E_2 = E_2(r, \omega)$ that*

$$|f^{(r)}(0)| \leqslant E_1 \, \|f\|_{\mathbb{L}_1[0,1]} + E_2. \tag{1.14}$$

Proof. Let Y_r be the perfect spline from Proposition 1.7. Then,

$$\|f\|_{\mathbb{L}_1[0,1]} \geqslant \int_0^1 f(x)\, Y_r^{(r)}(x)\, dx = (-1)^r \int_0^1 f^{(r)}(x)\, Y_r(x)\, dx$$

$$= (-1)^r \int_0^1 [f^{(r)}(x) - f^{(r)}(0)]\, Y_r(x)\, dx + (-1)^r f^{(r)}(0) \int_0^1 Y_r(x)\, dx. \tag{1.15}$$

Thus, using the inclusion $f^{(r)} \in H^\omega[0,1]$ and the inequality (1.15), we obtain the estimate (1.14) with $E_1 := \|Y_r\|^{-1}_{\mathbb{L}_1[0,1]}$ and $E_2 := \|Y_r\|^{-1}_{\mathbb{L}_1[0,1]} \cdot \|\omega \cdot Y_r\|_{\mathbb{C}[0,1]}$. ∎

PROPOSITION 1.9. *Let $A > 1$, ω be a concave modulus of continuity, and the function $f(t) \in \mathbb{AC}^{r+1}[0,A]$ be endowed with the properties*

$$\begin{aligned} &\text{(i)} \quad f^{(r)}(t) > 0, \qquad \textit{for a.e.} \quad t \in [0,A]; \\ &\text{(ii)} \quad f^{(r-1)} \in H^\omega[0,1]; \\ &\text{(iii)} \quad f^{(r)}(t) = 1, \qquad t \in [1,A]; \\ &\text{(iv)} \quad f \textit{ has } r \textit{ zeroes } \{\eta_l^0\}_{l=1}^r \textit{ satisfying inequalities} \\ &\qquad 0 \leqslant \eta_r^0 < \eta_{r-1}^0 < \cdots < \eta_2^0 \leqslant 1 < \eta_1^0 := A. \end{aligned} \tag{1.16}$$

Then, for each $k = 1, ..., r-1$, the derivative $f^{(k)}$ has precisely $r-k$ simple zeroes $\{\eta_l^k\}_{l=1}^{r-k}$,

$$0 \leqslant \eta_{r-k}^k < \eta_{r-k-1}^k < \cdots < \eta_2^k < \eta_1^k, \tag{1.17}$$

and there exist constants $E_{r,k}$ such that

$$\eta_1^k > E_{r,k} A > 1, \qquad 0 \leqslant k \leqslant r-1, \tag{1.18}$$

for all sufficiently large A's.

Proof. By the property (1.16), (iv), $f^{(k)}$ has *at least* $r-k$ distinct zeroes on $[0,A]$ for $1 \leqslant k \leqslant r-1$. On the other hand, by (1.16), (i), $f^{(r-1)}$ is monotone on $[0,A]$. Thus, $f^{(k)}$ can have *at most* $r-k$ zeroes counting multiplicities. Therefore, Rolle's theorem implies that $f^{(k)}$ has precisely $r-k$ simple zeroes $\{\eta_i^k\}_{i=1}^{r-k}$ enumerated in the decreasing order as in (1.17). We also observe that by (1.16), (iv) and Rolle's theorem, zeroes $\{\eta_i^k\}_{i=2}^{r-k}$ lie on the interval $[0,1]$.

The verification of the property (1.18) of rightmost zeroes $\{\eta_1^k\}_{k=0}^{r-1}$ proceeds by induction.

For $k=0$ the statement is true by (1.16), (iv): $\eta_1^0 := A$.

Suppose that we have proved the statement for any $k=0, ..., n \leqslant r-2$, i.e.,

$$\eta_1^k > E_{r,k} \cdot A > 1, \qquad k=0, ..., n. \tag{1.19}$$

Let us prove the property (1.18) for $k=n+1$.

First, $f^{(n)}(\eta_2^n) = f^{(n)}(\eta_1^n) = 0$. Therefore,

$$0 = \int_{\eta_2^n}^{\eta_1^n} f^{(n+1)}(\xi)\, d\xi = \int_{\eta_2^n}^{\eta_1^{n+1}} f^{(n+1)}(\xi)\, d\xi + \int_{\eta_1^{n+1}}^{\eta_1^n} f^{(n+1)}(\xi)\, d\xi, \tag{1.20}$$

where η_1^{n+1} is the rightmost zero of $f^{(n+1)}(t)$. By Rolle's theorem, it lies between η_1^n and η_2^n.

By Rolle's theorem, the other $r-n-2$ zeroes belong to the interval $[0, \eta_2^n]$. Thus, the function $f^{(n+1)}(t)$ changes its sign on the interval $[\eta_2^n, \eta_1^n]$ only at the point η_1^{n+1}.

Therefore, we can infer from (1.20) that

$$\int_{\eta_2^n}^{\eta_1^{n+1}} |f^{(n+1)}(\xi)|\, d\xi = \int_{\eta_1^{n+1}}^{\eta_1^n} |f^{(n+1)}(\xi)|\, d\xi = \tfrac{1}{2} \, \|f^{(n+1)}\|_{\mathbb{L}_1[\eta_2^n, \eta_1^n]}. \tag{1.21}$$

Let

$$I_k := \min_{p \in P_k} \|p\|_{\mathbb{L}_1[0,1]}, \qquad k \in N, \tag{1.22}$$

where P_k is the space of all polynomials of degree k with the leading coefficient $\pm 1/k!$. Then,

$$\min_{p \in P_k} \|p\|_{\mathbb{L}_1[a,b]} = (b-a)^{k+1} I_k. \tag{1.23}$$

By (1.16), (iii), $f^{(r)}(t) = 1$, $t \in [1, A]$, i.e., $f^{(n+1)}(t)$ is a polynomial of degree $r-n-1$ on the interval $[1, A]$ with the leading coefficient $1/(r-n-1)!$. Thus, by (1.23),

$$\|f^{(n+1)}\|_{\mathbb{L}_1[1, \eta_1^n]} \geqslant I_{r-n-1} (\eta_1^n - 1)^{r-n}. \tag{1.24}$$

Consecutively using the equalities (1.21), the inclusion $\eta_2^n \in [0, 1]$, the inequality (1.24), and the inductive assumption (1.19) for $k = n$, we derive the estimates

$$\int_{\eta_2^n}^{\eta_1^{n+1}} |f^{(n+1)}(\xi)|\, d\xi = \tfrac{1}{2}\, \|f^{(n+1)}\|_{\mathbb{L}_1[\eta_2^n, \eta_1^n]} \geqslant \tfrac{1}{2}\, \|f^{(n+1)}\|_{\mathbb{L}_1[1, \eta_1^n]}$$

$$\geqslant \tfrac{1}{2}\, I_{r-n-1}(\eta_1^n - 1)^{r-n} \geqslant \tfrac{1}{2}\, I_{r-n-1}(E_{r,n} \cdot A - 1)^{r-n}. \tag{1.25}$$

In a more compact form,

$$\|f^{(n+1)}\|_{\mathbb{L}_1[\eta_2^n, \eta_1^{n+1}]} \geqslant \tfrac{1}{2}\, I_{r-n-1}(E_{r,n} \cdot A - 1)^{r-n}. \tag{1.26}$$

We have already shown that each of the functions $f^{(k)}$ has a zero on $[0, A]$ for $k = 0, ..., r-1$. Thus, applying Proposition 1.6 and using properties (1.16), (ii), (iii) of the function f, we obtain the inequality

$$\|f^{(n+1)}\|_{\mathbb{C}[0,A]} \leqslant A^{r-n-2}\, \|f^{(r-1)}\|_{\mathbb{C}[0,A]} \leqslant A^{r-n-2}(A + \omega(1)) \leqslant 2A^{r-n-1}, \tag{1.27}$$

for $A > \omega(1)$. Consequently,

$$\|f^{(n+1)}\|_{\mathbb{L}_1[\eta_2^n, \eta_1^{n+1}]} \leqslant \|f^{(n+1)}\|_{\mathbb{C}[0,A]}\, (\eta_1^{n+1} - \eta_2^n) \leqslant 2A^{r-n-1}(\eta_1^{n+1} - \eta_2^n). \tag{1.28}$$

Combining the estimates (1.26) from below and (1.28) from above for the integral norm of the function $f^{(n+1)}(t)$, we derive the following estimate for the length of the interval $[\eta_2^n, \eta_1^{n+1}]$:

$$\eta_1^{n+1} - \eta_2^n \geqslant \tfrac{1}{4} I_{r-n-1} A^{n-r+1}(E_{r,n} \cdot A - 1)^{r-n}. \tag{1.29}$$

Due to the inclusion $\eta_2^n \in [0, 1]$, the inequality (1.29) implies that for all sufficiently large A's,

$$\eta_1^{n+1} \geqslant \tfrac{1}{4} I_{r-n-1} A^{n-r+1}(E_{r,n} \cdot A - 1)^{r-n} - 1 \geqslant E_{r,n+1} A > 1, \tag{1.30}$$

with the constant $E_{r,n+1}$ depending only on r, n. ∎

In conclusion, we state the properties of generating kernels $K(t)$ and $F(t)$ (see, e.g., [8]).

Let $r, m\colon 0 < m \leqslant r$, be integers.

Let $\{\tau_i\}_{i=0}^r$ be such that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_r \leqslant 1. \tag{1.31}$$

Derive $\{\alpha_i = \alpha_i(\tau_0, ..., \tau_r, m)\}_{i=0}^{r}$ from the system of linear equations

$$\sum_{i=0}^{r} \alpha_i \tau_i^k = \delta_{m,k}, \qquad k = 0, ..., r. \tag{1.32}$$

Put

$$\begin{aligned} K(u) &= -\frac{1}{(r-1)!} \sum_{i=1}^{r} \alpha_i (\tau_i - u)_+^{r-1}, \\ F(u) &= \frac{1}{r!} \sum_{i=1}^{r} \alpha_i (\tau_i - u)_+^{r}. \end{aligned} \tag{1.33}$$

PROPOSITION 1.10. *Let* $r, m \in \mathbb{N} : 0 < m \leqslant r$, *the points* $\{\tau_i\}_{i=0}^{r}$ *be as in* (1.31), *the coefficients* $\{\alpha_i\}_{i=0}^{r}$ *be defined in* (1.32), *and the kernels* $K(t)$, $F(t)$ *be defined by* (1.33). *Then,*

I. sign $\alpha_i = (-1)^{i+m}$, $i = 0, ..., r$;

II. *for* $0 < m < r$, *the kernel* $F(t)$ *is simple on* $[0, \tau_r]$, *and for some* $c \in (0, 1)$,

$$\operatorname{sign} K(t) = (-1)^{r+m}, \ t \in (0, c); \qquad \operatorname{sign} K(t) = (-1)^{r+m+1}, \quad t \in (c, 1);$$

III. *for* $m = r$, $K(t)$ *does not change the sign*: $K(t) < 0$, $t \in [0, 1)$.

2. CONSTRUCTION OF ZOLOTAREV ω-POLYNOMIALS

2.1. *Sufficient Conditions of Extremality in the Kolmogorov–Landau Problem*

Fix $r, m \in \mathbb{N} : 0 < m \leqslant r$. Let $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots \tau_r := b \leqslant 1$.

Let $\{\alpha_i\}_{i=0}^{r}$ be the solutions of the following system of linear equations:

$$\sum_{i=0}^{r} \alpha_i \tau_i^k = \delta_{m,k}, \qquad k = 0, ..., r. \tag{2.1}$$

In (2.1), we follow the convention $[\tau_0]^0 = 0^0 := 1$. Put

$$\begin{aligned} K(u) &= -\frac{1}{(r-1)!} \sum_{i=0}^{r} \alpha_i (\tau_i - u)_+^{r-1}, \\ F(u) &= \frac{1}{r!} \sum_{i=0}^{r} \alpha_i (\tau_i - u)_+^{r}. \end{aligned} \tag{2.2}$$

Fix $f \in W^rH^\omega[0, 1]$. The Taylor's formula reads

$$f(\tau) = \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} \tau^k + \frac{1}{(r-1)!} \int_0^1 f^{(r)}(u)(\tau - u)_+^{r-1} \, du, \qquad 0 \leqslant \tau \leqslant 1. \tag{2.3}$$

We distinguish two cases in deriving the formula for $f^{(m)}(0)$.

Case 1. $0 < m < r$.

From (2.1)–(2.3) we find the formula for the value of the mth derivative $f^{(m)}$ at the origin:

$$f^{(m)}(0) = m! \sum_{i=0}^{r} \alpha_i f(\tau_i) + m! \int_0^b f^{(r)}(u) \, K(u) \, du. \tag{2.4}$$

By Proposition 1.10, the kernel $F(t) = \int_1^t K(y) \, dy$ is *simple* in the sense of Definition 1.1. Therefore, by Korneichuk's Lemma 1.2,

$$\sup_{h \in H^\omega[0, b]} \int_0^b h(t) \, K(t) \, dt = \sup_{h \in H_0^\omega[0, b]} \int_0^b h(t) \, K(t) \, dt = \int_0^b \Re(F; t) \, \omega'(t) \, dt, \tag{2.5}$$

where the classes $H_0^\omega[a, b]$ are defined in (0.12), and the rearrangements $\Re(\Psi; t)$ of simple kernels Ψ are introduced in (1.4). The Korneichuk lemma also provides the formula for the derivative of the function $h^*(t)$ realizing the supremum in (2.5):

$$\frac{d}{dt} h^*(t) = \begin{cases} (-1)^{r+m+1} \, \omega'(\rho(t) - t), & 0 \leqslant t \leqslant c, \\ (-1)^{r+m+1} \, \omega'(t - \rho^{-1}(t)), & c \leqslant t \leqslant \tau_r; \end{cases} \tag{2.6}$$

where c is a unique zero of the kernel $K(t)$ on the open interval $(0, b)$, and the function $\rho\colon [0, c] \to [c, b]$ is derived from the equations

$$F(t) = F(\rho(t)), \qquad 0 \leqslant t \leqslant c. \tag{2.7}$$

From (2.4) and (2.5) we obtain the estimate

$$|f^{(m)}(0)| \leqslant m! \left(\sum_{i=0}^{r} |\alpha_i| \right) \|f\|_{\mathbb{C}[0,1]} + m! \int_0^b \Re(F; t) \, \omega'(t) \, dt. \tag{2.8}$$

Case 2. $m = r$.

In this case, by (2.1), $\int_0^b K(u) \, du = -1/r! \sum_{i=0}^{r} \alpha_i \tau_i^r = -1/r!$. Therefore, by (2.1)–(2.3),

$$f^{(r)}(0) = r! \sum_{i=0}^{r} \alpha_i f(\tau_i) + r! \int_0^b [f^{(r)}(u) - f^{(r)}(0)] \, K(u) \, du. \tag{2.9}$$

Notice that $f^{(r)}(x)-f^{(r)}(0)\in H_0^{\omega}[0,1]$, if $f\in W^rH^{\omega}[0,1]$. In Proposition 1.10 we showed that $K(u)<0$, $0\leqslant u<b$. Therefore,

$$\sup_{h\in H_0^{\omega}[0,1]}\int_0^b h(u)\,K(u)\,du=-\int_0^b \omega(u)\,K(u)\,du=\int_0^b \omega'(u)\,F(u)\,du. \tag{2.10}$$

The equality sign is attained in (2.10), if and only if

$$f^{(r)}(u)-f^{(r)}(0)=-\omega(u), \qquad 0\leqslant u\leqslant b. \tag{2.11}$$

Consequently, by (2.9) and (2.10),

$$|f^{(r)}(0)|\leqslant r!\left(\sum_{i=0}^{r}|\alpha_i|\right)\|f\|_{\mathbb{C}[0,1]}+r!\int_0^b \omega'(t)\,F(t)\,dt. \tag{2.12}$$

By Proposition 1.10, in both cases $(-1)^{i+m}\alpha_i>0$, $i=0,\dots,r$. Combining these two cases and taking into account our observation (2.5), we give sufficient conditions for a function $f\in W^rH^{\omega}[0,1]$ to realize the equality sign in inequalities (2.8) for $0<m<r$ and (2.12) for $m=r$:

$$\begin{aligned}&\text{(i)}\quad f(\tau_i)=(-1)^{i+m}\|f\|_{\mathbb{C}[0,b]}, \qquad i=0,\dots,r;\\ &\text{(ii)}\quad \sup_{h\in H_0^{\omega}[0,b]}\int_0^b h(x)\,K(x)\,dx=\int_0^b [f^{(r)}(x)-f^{(r)}(0)]\,K(x)\,dx.\end{aligned} \tag{2.13}$$

Therefore, the problem is to choose the collection of points $\{\tau_i\}_{i=0}^r$ *simultaneously* endowed with two properties: $\{\tau_i\}_{i=0}^r$ are *the knots* of the generating kernel K for the function $f^{(r)}(x)$ and *the alternance points* of the function f on the interval $[0,b]$.

2.2. *Characterization of Zolotarev ω-Polynomials*

THEOREM 2.1. *Let $r, m\in\mathbb{N}: 0<m\leqslant r$, and $B>0$. There exists a set of points $\{\tau_i(B)=\tau_i(B,r,m,\omega)\}_{i=0}^r$, $0=\tau_0(B)<\tau_1(B)<\cdots<\tau_r(B)\leqslant 1$, and the function $Z_B=Z_{B,r,m,\omega}\in W^rH^{\omega}[0,1]$ with the properties*

$$\text{(i)}\qquad \sup_{h\in H_0^{\omega}[0,\tau_r(B)]}\int_0^{\tau_r(B)} h(t)\,K_B(t)\,dt=\int_0^{\tau_r(B)}[Z_B^{(r)}(t)-Z_B^{(r)}(0)]\,K_B(t)\,dt,$$

where the kernel K_B is defined by (2.1), (2.2) for $\{\tau_i=\tau_i(B)\}_{i=0}^r$;

(ii) $Z_B(\tau_i(B)) = (-1)^{i+m} \|Z_B\|_{\mathbb{C}[0,\tau_r(B)]^z} = (-1)^{i+m} B, \qquad i=0, ..., r;$ (2.14)

(iii) *if* $\tau_r(B) < 1$, *then* $\frac{d}{dt} Z_B(\tau_r(B)) = 0.$

Proof. Fix $A > 4$ and ε, $0 < \varepsilon < 1/r$. Let

$$\mathbb{S}^r_A := \left\{ s = (s_1, ..., s_{r+1}) \in \mathbb{R}^{r+1} \,\middle|\, \sum_{i=1}^{r+1} |s_i| = A \right\}. \tag{2.15}$$

We generate collections of points $\{t_j = t_j(s)\}_{j=0}^{r+1}$, $\{\bar{t}_j = \bar{t}_j(s)\}_{j=0}^{r+1}$, $\{T_j = T_j(s)\}_{j=0}^{r}$, and $\{\tau_j = \tau_j(s)\}_{j=0}^{r}$:

$$\begin{aligned}
&t_0(s) = 0, && t_j(s) = \sum_{i=1}^{j} |s_i|, && j = 1, ..., r+1;\\
&\bar{t}_0(s) = 0, && \bar{t}_j(s) = \min\{t_j(s), 1\}, && j = 1, ..., r+1;\\
&T_0(s) = 0, && T_j(s) = \frac{t_j(s) + \varepsilon j}{1 + \varepsilon r}, && j = 1, ..., r;\\
&\tau_0(s) = 0, && \tau_j(s) = \frac{\bar{t}_j(s) + \varepsilon j}{1 + \varepsilon r}, && j = 1, ..., r.
\end{aligned} \tag{2.16}$$

By (2.16),

$$\tau_i(s) - \tau_{i-1}(s) \geqslant \frac{\varepsilon}{1+\varepsilon r}, \qquad T_i(s) - T_{i-1}(s) \geqslant \frac{\varepsilon}{1+\varepsilon r}, \qquad i = 1, ..., r, \quad s \in \mathbb{S}^r_A, \tag{2.17}$$

and

$$|\bar{t}_i(s) - \tau_i(s)| \leqslant \varepsilon r, \qquad |T_i(s) - t_i(s)| < \varepsilon r A, \qquad i = 0, ..., r, \quad s \in \mathbb{S}^r_A. \tag{2.18}$$

Also by (2.16), the points $\{\tau_i(s)\}_{i=0}^{r}$ belong to the interval $[0, 1]$:

$$0 \leqslant \tau_i(s) \leqslant \frac{\bar{t}_i(s) + \varepsilon i}{1 + \varepsilon r} \leqslant \frac{1 + \varepsilon i}{1 + \varepsilon r} \leqslant 1, \qquad i = 1, ..., r, \quad s \in \mathbb{S}^r_A.$$

Let $\{\alpha_i(s) = \alpha_i(s, r, m, \varepsilon)\}_{i=0}^{r}$ satisfy the system of linear equations

$$\sum_{i=0}^{r} \alpha_i [\tau_i(s)]^k = \delta_{m,k}, \qquad k = 0, ..., r. \tag{2.19}$$

As before, in (2.19) we follow the convention $[\tau_0(s)]^0 = 0^0 := 1$.

Let us introduce kernels K_s and F_s:

$$K_s(t) = -\frac{1}{(r-1)!} \sum_{0=1}^{r} \alpha_i(s)(t-\tau_i(s))_+^{r-1};$$
$$F_s(t) = \frac{1}{r!} \sum_{0=1}^{r} \alpha_i(s)(t-\tau_i(s))_+^{r}. \tag{2.20}$$

Let the function $f_s \in H_0^\omega[0, \tau_r(s)]$ be extremal in the problem

$$\int_0^{\tau_r(s)} h(t)\, K_s(t)\, dt \to \sup, \qquad h \in H_0^\omega[0, \tau_r(s)]. \tag{2.21}$$

If $0 < m < r$, by Lemma 1.2, the derivative $(d/dx)\, f_s(x)$ is expressed by the formula

$$\frac{d}{dt} f_s(t) = \begin{cases} (-1)^{r+m+1}\, \omega'(\rho_s(t) - t), & 0 \leqslant t \leqslant c(s), \\ (-1)^{r+m+1}\, \omega'(t - \rho_s^{-1}(t)), & c(s) \leqslant t \leqslant \tau_r(s), \end{cases} \tag{2.22}$$

where $c(s)$ is a unique zero of K_s on the open interval $(0, \tau_r(s))$, and the function $\rho_s\colon [0, c(s)] \to [c(s), \tau_r(s)]$ is derived from the equations

$$F_s(t) = F_s(\rho_s(t)), \qquad 0 \leqslant t \leqslant c(s).$$

According to (2.11), for $m = r$ we put

$$f_s(t) = -\omega(t), \qquad 0 \leqslant t \leqslant \tau_r(s). \tag{2.23}$$

The extension $g_s(t)$ of $f_s(t)$ from $[0, \tau_r(s)]$ to $[0, A]$ is defined by the formula

$$g_s(t) = \begin{cases} f_s(t), & 0 \leqslant t \leqslant \tau_r(s), \\ (-1)^{r+m+1}\, [\omega(\tau_r(s)) + t - \tau_r(s)], & \tau_r(s) \leqslant t \leqslant A. \end{cases} \tag{2.24}$$

Notice that by the definitions (2.22) and (2.24), the function $g_{s*}(t)$ is monotone on $[0, A]$, and

$$(-1)^{r+m+1} \frac{d}{dt} g_s(t) > 0, \qquad \text{for a.e.} \quad t \in [0, A]. \tag{2.25}$$

Also by (2.22) and (2.24),

$$g_s \in H^{\tilde\omega}[0, A], \qquad \tilde\omega(t) := \omega(t) + t, \quad t \in \mathbb{R}_+. \tag{2.26}$$

Let us introduce the function

$$V_s(x) = \frac{1}{(r-2)!} \int_0^A (x-t)_+^{r-2}\, g_s(t)\, dt, \qquad 0 \leqslant x \leqslant A. \tag{2.27}$$

Notice that $V_s^{(r-1)}(x) = g_s(x)$, $0 \leqslant x \leqslant A$, and $V_s^{(i)}(0) = 0$, $i = 0, ..., r-1$.

Let $q_s(t)$ be the polynomial of degree $r-1$ interpolating $V_s(t)$ at r distinct points $\{T_i(s)\}_{i=1}^r$:

$$q_s(T_i(s)) = V_s(T_i(s)), \qquad i = 1, ..., r. \tag{2.28}$$

Put

$$W_s(x) := V_s(x) - q_s(x), \qquad 0 \leqslant x \leqslant A. \tag{2.29}$$

By (2.28),

$$W_s(T_i(s)) = 0, \qquad i = 1, ..., r. \tag{2.30}$$

By our observation (2.25), sign $W_s^{(r)}(t) = \operatorname{sign}(d/dt)\, g_s(t) = (-1)^{r+m+1}$ for a.e. $t \in [0, A]$. Thus, by the Rolle's theorem, all zeroes $\{T_i(s)\}_{i=1}^r$ are simple, and

$$\operatorname{sign} W_s(t) = (-1)^{i+m}, \qquad t \in (T_{i-1}(s), T_i(s)), \quad i = 1, ..., r+1. \tag{2.31}$$

Put

$$\Delta_i(s) := \int_{\bar{t}_{i-1}(s)}^{\bar{t}_i(s)} |W_s(t)|\, dt, \qquad i = 1, ..., r. \tag{2.32}$$

Next, for $s = (s_1, ..., s_{r+1}) \in \mathbb{S}_A^r$, we define the function U_s on the interval $[0, 1]$:

$$U_s(t) = (-1)^{i+m} [\operatorname{sign} s_i]\, |W_s(t)|, \qquad \bar{t}_{i-1}(s) \leqslant t \leqslant \bar{t}_i(s), \quad i = 1, ..., r+1. \tag{2.33}$$

Let

$$\tilde{H}_s(t) = \int_0^t U_s(x)\, dx, \qquad 0 \leqslant t \leqslant 1. \tag{2.34}$$

Let us introduce the constants

$$C := A^{r-1}[A + \omega(A)], \qquad \Theta = \min\left\{1, rB\left[C + \frac{4rB}{A}\right]^{-1}\right\}. \tag{2.35}$$

Put

$$\theta_s := \max\{\Theta, \bar{t}_r(s)\}, \qquad s \in \mathbb{S}^r_A. \tag{2.36}$$

We introduce the polynomial $p_s(t) = \sum_{i=0}^{r-1} a_i(s)\, t^i$ of the best approximation for the function $\tilde{H}_s(t)$ on the interval $[0, \theta_s]$:

$$\|\tilde{H}_s - p_s\|_{\mathbb{C}[0,\theta_s]} = \min_{p \in \mathcal{P}_{r-1}} \|\tilde{H}_s - p\|_{\mathbb{C}[0,\theta_s]}, \tag{2.37}$$

where $\mathcal{P}_{r-1}$ is the space of polynomials of degree at most $r-1$. Put

$$H_s(t) = \tilde{H}_s(t) - p_s(t), \qquad t \in [0, 1], \tag{2.38}$$

and

$$D(s) := \sum_{i=1}^{r} [\operatorname{sign} s_i]\, \Delta_i(s) - \frac{2rB}{A} \sum_{i=1}^{r+1} s_i. \tag{2.39}$$

The mapping $\kappa\colon \mathbb{S}^r_A \to \mathbb{R}^r$ is defined as

$$\kappa(s) = (a_1(s), ..., a_{r-1}(s), D(s)), \qquad s \in \mathbb{S}^r_A. \tag{2.40}$$

In the following lemma we prove the continuity of the mapping κ.

LEMMA 2.1.1. *Let the mapping κ on the sphere $\mathbb{S}^r_A$ be defined in* (2.40). *Then, the mapping $s \mapsto \kappa(s)$, $s \in \mathbb{S}^r_A$, is continuous.*

Proof. From inequalities (2.17) it follows that the Vandermonde determinant of the system of linear equations (2.19) never vanishes on $\mathbb{S}^r_A$. The Kramer's formula for the solution of (2.19) coupled with the continuity of the mapping $s \mapsto \{\tau_i(s)\}_{i=0}^{r+1}$ implies the continuous dependence of coefficients $\{\alpha_i(s)\}_{i=0}^{r}$ on s.

Also by (2.19), $\tau_r(s) \geqslant r/(1+\varepsilon r) = =: d$, $s \in \mathbb{S}^r_A$.

Let us introduce the dylated version of the kernel K_s:

$$\hat{K}_s(t) := K_s\left(\frac{\tau_r(s)\, t}{d}\right), \qquad 0 \leqslant t \leqslant d. \tag{2.41}$$

The continuity of the mapping $s \mapsto (\{\alpha_i(s)\}_{i=0}^{r}, \tau_r(s))$ implies the continuous dependence of the family of kernels $\{\hat{K}_s\}_{s \in \mathbb{S}^r_A}$ on s in the metrics $\mathbb{L}_1[0, d]$ (and even $\mathbb{C}[0, d]$). Therefore, we can apply Lemma 1.3 to the family of kernels $K_s(t)$ and functions $f_s(t)$ extremal in the problem (2.21). Then, Lemma 1.3 and the definition (2.24) of the extension $g_s(t)$ guarantee the continuity of the mapping $s \mapsto g_s$ in $\mathbb{C}[0, A]$.

Then, by the definition (2.27), the mapping $s \mapsto V_s$ is continuous in $\mathbb{C}[0, A]$. From the separation property (2.17) of the points $\{T_i(s)\}_{i=1}^r$ and the Lagrange formula for the interpolating polynomial $q_s(t)$ we deduce the continuity of the mapping $s \mapsto W_s$ in $\mathbb{C}[0, A]$. In particular, there exists such a constant $\mathcal{M}$ that

$$\|U_s\|_{\mathbb{L}_\infty[0,1]} = \|W_s\|_{\mathbb{C}[0,1]} \leqslant \mathcal{M}, \qquad \text{for all} \quad s \in \mathbb{S}_A^r. \tag{2.42}$$

Next, using the definitions (2.33) of the function U_s and (2.34) of the function $\tilde{H}_s(t)$, we derive the chain of inequalities

$$\begin{aligned}\|\tilde{H}_{s_1} - \tilde{H}_{s_2}\|_{\mathbb{C}[0,1]} &\leqslant \|U_{s_1} - U_{s_2}\|_{\mathbb{L}_1[0,1]} \\ &\leqslant 2(\mathcal{M}\, \|s_1 - s_2\|\, l_1^{r+1} + \|W_{s_1} - W_{s_2}\|_{\mathbb{L}_1[0,1]}),\end{aligned} \tag{2.43}$$

which proves the continuity of the mapping $s \mapsto \tilde{H}_s$ in $\mathbb{C}[0,1]$. Then, the continuous dependence on s of coefficients $\{a_i(s)\}_{i=0}^r$ of the polynomial p_s of the best approximation for $\tilde{H}_s$ on $[0, \theta_s]$ follows from the uniqueness of p_s and separation of the length θ_s of the interval $[0, \theta_s]$ from zero: $\theta_s \geqslant \Theta$, $s \in \mathbb{S}_A^r$.

It remains to prove the continuity of the mapping $s \mapsto D(s)$ defined in (2.39), (2.32). The following proposition accomplishes this objective.

PROPOSITION 2.1.2. *For each* $s = (s_1, s_2, ..., s_{r+1}) \in \mathbb{S}_A^r$, *let* $\{\Delta_i(s)\}_{i=1}^r$ *be defined in* (2.32). *Then, for* $i = 1, ..., r$,

$$\Delta_i(s) \to 0, \qquad \textit{as} \quad s_i \to 0. \tag{2.44}$$

Proof. By (2.30), W_s has r distinct zeroes $\{T_i(s)\}_{i=1}^r$ on the interval $[0, A]$. Therefore, by the Rolle's theorem, the derivative $W_s^{(k)}$ has a zero on $[0, A]$ for $k = 0, ..., r-1$. Recall that by (2.29), (2.27), $W_s^{(r-1)}(t) = g_s(t) + \alpha(s)$, $0 \leqslant t \leqslant 1$, where $\alpha(s) := q_s^{(r-1)}(t)$. Then, applying Proposition 1.6 and taking into account the inclusion (2.26), we infer that for $k = 1, ..., r-1$,

$$\begin{aligned}\|W_s\|_{\mathbb{C}[0,A]} &\leqslant \cdots \leqslant A^k\, \|W_s^{(k)}\|_{\mathbb{C}[0,A]} \leqslant \cdots \leqslant A^{r-1}\, \|W_s^{(r-1)}\|_{\mathbb{C}[0,A]} \\ &\leqslant A^{r-1}(A + \omega(A)) =: C.\end{aligned} \tag{2.45}$$

Thus, by the definition (2.32) and (2.45),

$$\Delta_i(s) = \|W_s\|_{\mathbb{L}_1[\bar{t}_{i-1}(s),\, \bar{t}_i(s)]} \leqslant \|W_s\|_{\mathbb{C}[0,A]}\, |s_i| \leqslant C\, |s_i|, \qquad i = 1, ..., r. \tag{2.46}$$

The estimates (2.46) imply (2.44). ∎

Proposition 2.1.2 completes the proof of continuity of the mapping $s \mapsto \kappa(s)$. ∎

From the definitions (2.34) of $\tilde{H}_s$ and (2.39) of $D(s)$ one can readily observe that the mapping $s \mapsto \kappa(s)$ is odd: $\kappa(-s) = -\kappa(s)$, $s \in \mathbb{S}^r_A$. An application of the Borsuk's theorem (Theorem 1.4) to the mapping κ guarantees the existence of a point satisfying the equation $\kappa(s) = 0$, $s \in \mathbb{S}^r_A$, or equivalently,

$$D(s) = 0, \qquad a_i(s) = 0, \qquad i = 1, ..., r, \qquad s \in \mathbb{S}^r_A. \tag{2.47}$$

Fix a solution s^* of the equation (2.47) and put

$$\begin{aligned} t_i^* &= t_i(s^*), \qquad \bar{t}_i^* = \bar{t}_i(s^*), \qquad i = 1, ..., r+1; \\ \tau_i^* &= \tau_i(s^*), \qquad T_i^* = T_i(s^*), \qquad i = 1, ..., r. \end{aligned} \tag{2.48}$$

LEMMA 2.1.3. *Let* $\{\theta_s\}_{s \in \mathbb{S}^r_A}$ *be introduced in* (2.36). *Then,*

$$\theta_{s^*} = \bar{t}_r^*. \tag{2.49}$$

Proof. Suppose, on the contrary, that $\bar{t}_r^* < \theta_{s^*}$. By the definition (2.35), in this case, $\theta_{s^*} = \Theta \leqslant 1$, and $\bar{t}_r^* := \min\{t_r^*, 1\} = t_r^*$.

Let $s^* = (s_1^*, ..., s_{r+1}^*)$. From the equation $D(s^*) = 0$ it follows that

$$\sum_{i=1}^r [\text{sign}\, s_i^*]\, \Delta_i(s^*) - \frac{2rB}{A} \sum_{i=1}^r s_i^* = \frac{2rB}{A} s_{r+1}^*. \tag{2.50}$$

Let us estimate the left-hand side of this equation from above and the right-hand side from below.

By (2.46) and our assumption $t_r^* = \bar{t}_r^* = \sum_{i=1}^r |s_i^*| < \Theta$, we have

$$\left| \sum_{i=1}^r \text{sign}\, s_i^*\, \Delta_i(s^*) - \frac{2rB}{A} \sum_{i=1}^r s_i^* \right| \leqslant \left(C + \frac{2rB}{A} \right) \sum_{i=1}^r |s_i^*| < \left(C + \frac{2rB}{A} \right) \Theta. \tag{2.51}$$

On the other hand,

$$|s_{r+1}^*| = t_{r+1}^* - t_r^* = A - t_r^* > A - \Theta. \tag{2.52}$$

Combining Eq. (2.50) and inequalities (2.51) and (2.52), we conclude that

$$\left(C + \frac{2rB}{A} \right) \Theta > \frac{2rB}{A} (A - \Theta)$$

or

$$\Theta > 2rB \left(C + \frac{4rB}{A} \right)^{-1}. \tag{2.53}$$

This contradiction with the definition (2.35) of Θ shows that $\bar{t}_r^* > \Theta$, and

$$\theta_{s*} = \max\{\Theta, \bar{t}_r^*\} = \bar{t}_r^*. \quad \blacksquare \tag{2.54}$$

By the property (2.25), the function H_{s*} is not a polynomial of degree $r-1$. In particular, $\|H_{s*}\|_{\mathbb{C}[0,\bar{t}_r^*]} > 0$.

The Chebyshev theorem (Theorem 1.5) implies that the function $H_s(t)$ has $r+1$ points of alternance $\{z_i\}_{i=0}^r$, $0 \leqslant z_0 < z_1 < \cdots < z_r \leqslant \bar{t}_r^*$, on the interval $[0, \bar{t}_r^*] = [0, \theta_{s*}]$. Therefore, the derivative $(d/dt)\, H_{s*}(t)$ has *at least* $r-1$ points $\{z_i\}_{i=1}^r$ of sign change on the interval $[0, \bar{t}_r^*]$.

On the other hand, by the definition (2.38) and Eq. (2.47) for $s = s^*$,

$$\frac{d}{dt} H_{s*}(t) = \frac{d}{dt} \tilde{H}_{s*}(t) - \sum_{i=1}^{r-1} a_i(s^*)\, i t^{i-1} = U_{s*}(t), \qquad 0 \leqslant t \leqslant 1. \tag{2.55}$$

By the definition (2.33), the function $U_{s*}(t) = (d/dt)\, H_{s*}(t)$ can have *at most* $r-1$ points $\{\bar{t}_i^*\}_{i=1}^{r-1}$ of sign change on the interval $[0, \bar{t}_r^*]$. This argument shows that $(d/dt)\, H_{s*}(t)$ has precisely $r-1$ points of sign change on $[0, \bar{t}_r^*]$, and

$$z_i = \bar{t}_i^* = t_i^*, \qquad i = 1, ..., r-1, \qquad z_0 = 0, \qquad z_r = \bar{t}_r^*. \tag{2.56}$$

Thus, for $\chi \in \{-1, 1\}$,

$$H_{s*}(\bar{t}_i^*) = (-1)^i \chi\, \|H_{s*}\|_{\mathbb{C}[0,\bar{t}_r^*]}, \qquad i = 0, ..., r. \tag{2.57}$$

The mappings $s \mapsto \kappa(s)$ and $s \mapsto H_s$ are odd. Therefore, we can assume without loss of generality (if necessary, considering $-s^*$) that $\chi = (-1)^m$ in (2.57). By (2.57) with $\chi = (-1)^m$ and the definition (2.32) of $\{\Delta_i(s)\}_{i=1}^r$, we have

$$\begin{aligned} 2(-1)^{i+m} \|H_{s*}\|_{\mathbb{C}[0,\bar{t}_r^*]} &= H_{s*}(\bar{t}_i^*) - H_{s*}(\bar{t}_{i-1}^*) \\ &= (-1)^{i+m} \operatorname{sign} s_i^* \int_{\bar{t}_{i-1}^*}^{\bar{t}_i^*} |W_{s*}(t)|\, dt \\ &= (-1)^{i+m} \operatorname{sign} s_i^*\, \Delta_i(s^*), \qquad i = 1, ..., r. \end{aligned} \tag{2.58}$$

Consequently, (2.58) and (2.55) with (2.33) lead us to the conclusion that

$$\begin{aligned} &\text{(A)} \quad \operatorname{sign} s_i^* = 1, \qquad i = 1, ..., r; \\ &\text{(B)} \quad \Delta_i(s^*) = 2\, \|H_{s*}\|_{\mathbb{C}[0,\bar{t}_r^*]}, \qquad i = 1, ..., r; \\ &\text{(C)} \quad \frac{d}{dt} H_{s*}(t) = (-1)^{i+m} |W_{s*}(t)|, \qquad \bar{t}_{i-1}^* \leqslant t \leqslant \bar{t}_i^*, \quad i = 1, ..., r+1. \end{aligned} \tag{2.59}$$

Our objective is to show that

$$\operatorname{sign} s_{r+1}^* = \operatorname{sign} s_{r+1}^*(A) = 1, \tag{2.60}$$

for all sufficiently large A's.

In order to accomplish this goal, we need to eliminate the other cases $\operatorname{sign} s_{r+1}^* = -1$ and $\operatorname{sign} s_{r+1}^* = 0$.

Let us assume that $\operatorname{sign} s_{r+1}^* = -1$. In this case, we can compute $D(s^*)$ using the properties (2.59) and the definition (2.39):

$$\begin{aligned} D(s^*) &= 2r \, \|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} - \frac{2rB}{A} t_r^* + \frac{2rB}{A}(A - t_r^*) \\ &= 2r \, \|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} - \frac{2rB}{A}(2t_r^* - A). \end{aligned} \tag{2.61}$$

Therefore, the equation $D(s^*) = 0$ and (2.61) imply that

$$\|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} = \frac{B}{A}(2t_r^* - A). \tag{2.62}$$

We derive two inequalities from (2.62),

$$t_r^*(A) \geqslant \frac{A}{2} \tag{2.63}$$

and, using the inclusion $t_r^* \in [0, A]$,

$$\|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} \leqslant B. \tag{2.64}$$

If we assume that $s_{r+1}^* = 0$, then

$$t_r^* := t_{r+1}^* - |s_{r+1}^*| = t_{r+1}^* := A. \tag{2.65}$$

Therefore,

$$D(s^*) = 2r\left(\|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} - \frac{B}{A} t_r^* \right) = 2r(\|H_{s^*}\|_{\mathbb{C}[0,\, \bar{t}_r^*]} - B) = 0. \tag{2.66}$$

Thus,

$$\|H_{s^*}\|_{\mathbb{C}[0,\, t_r^*]} = \|H_{s^*}\|_{\mathbb{C}[0,\, 1]} = B. \tag{2.67}$$

The pairwise combinations of properties (2.63) and (2.65), (2.64) and (2.67) lead us to the conclusion that we have two properties in the case sign $s^*_{r+1} \leqslant 0$:

$$\begin{aligned} &\text{(A)} \quad \|H_{s^*}\|_{\mathbb{C}[0,1]} \leqslant B; \\ &\text{(B)} \quad t^*_r \geqslant \frac{A}{2}. \end{aligned} \tag{2.68}$$

Notice that by (2.68), (B), and our choices $A > 4$ and $\varepsilon < 1/r$,

$$\begin{aligned} &\text{(i)} \quad \bar{t}^*_r = \min\{t^*_r, 1\} = 1, \qquad \text{and} \qquad \tau^*_r = \frac{\bar{t}^*_r + \varepsilon r}{1+\varepsilon r} = \frac{1+\varepsilon r}{1+\varepsilon r} = 1; \\ &\text{(ii)} \quad T^*_r = \frac{t^*_r + \varepsilon r}{1+\varepsilon r} \geqslant \frac{A/2 + \varepsilon r}{1+\varepsilon r} > \frac{A}{4} > 1. \end{aligned} \tag{2.69}$$

Let us show that the inequalities (2.68), (A) and (2.68), (B) are mutually incompatible.

Indeed, by the properties (2.59), the definition (2.32) and (2.68), (A),

$$\|W_{s^*}\|_{\mathbb{L}_1[0,1]} = \|W_{s^*}\|_{\mathbb{L}_1[0,\bar{t}^*_r]} = \sum_{i=1}^{r} \Delta_i(s^*) = 2r\,\|H_{s^*}\|_{\mathbb{C}[0,1]} \leqslant 2rB. \tag{2.70}$$

By (2.27), (2.29), and (2.69), (i)

$$W^{(r-1)}_{s^*}(t) = g_{s^*}(t) + \alpha(s^*) \in H^\omega[0, \tau^*_r] = H^\omega[0,1]. \tag{2.71}$$

Therefore, Proposition 1.8 provides two constants $\mathscr{E}_1 = \mathscr{E}_1(r)$ and $\mathscr{E}_2 = \mathscr{E}_2(r,\omega)$ such that

$$|W^{(r-1)}_{s^*}(0)| \leqslant \mathscr{E}_1\,\|W_{s^*}\|_{\mathbb{L}_1[0,1]} + \mathscr{E}_2 \leqslant 2r\mathscr{E}_1 B + \mathscr{E}_2. \tag{2.72}$$

On the other hand, by the definition (2.25) and the relations (2.69), (2.71), the function $W_{s^*}(t)$ is endowed with the properties

$$\begin{aligned} &\text{(i)} \quad (-1)^{r+m+1}\,W^{(r)}_{s^*}(t) = \frac{d}{dt}\,g_{s^*}(t) > 0, \qquad \text{for a.e.} \quad t \in [0, A]; \\ &\text{(ii)} \quad W^{(r-1)}_{s^*}(t) \in H^\omega[0,1]; \\ &\text{(iii)} \quad W^{(r)}_{s^*}(t) = (-1)^{r+m+1}, \qquad t \in [1, T^*_r]; \\ &\text{(iv)} \quad W_{s^*} \text{ has } r \text{ zeroes } \{T^*_i\}_{l=1}^{r}: \\ &\qquad 0 \leqslant T^*_1 < T^*_2 < \cdots < T^*_{r-1} \leqslant 1 < \frac{A}{4} < T^*_r. \end{aligned} \tag{2.73}$$

Applying Proposition 1.9 to the function $(-1)^{r+m+1} W_{s^*}(t)$ and using the property (2.69), (ii) we obtain the following estimate for the only zero η_1^{r-1} of the function $W_{s^*}^{(r-1)}$:

$$\eta_1^{r-1} > \mathscr{E}_3 T_r^* > \tfrac{1}{4}\mathscr{E}_3 A > 1, \qquad \text{for all} \quad A \geqslant A_0, \tag{2.74}$$

for some constant $\mathscr{E}_3 = \mathscr{E}_3(r, \omega)$ dependent only on r, ω, and some $A_0 > 0$.

However, by the definition (2.24) of the function $W_{s^*}^{(r-1)}$,

$$W_{s^*}^{(r-1)}(t) = g_{s^*}(t) + \alpha(s^*) = (-1)^{r+m+1}(t - \eta_1^r), \qquad t \in [1, A], \tag{2.75}$$

and $W_{s^*}^{(r-1)}(t)$ is monotone on the whole interval $[0, A]$. Therefore, by (2.75) and the estimate (2.74),

$$|W_{s^*}^{(r-1)}(0)| \geqslant |W_{s^*}^{(r-1)}(1)| = \eta_1^{r-1} - 1 \geqslant \tfrac{1}{4}\mathscr{E}_3 A - 1. \tag{2.76}$$

The juxtaposition of the estimates (2.72) and (2.76) for $|W_{s^*}(0)$ leads us to the conclusion that the inequalities become incompatible for all

$$A > \hat{A} := \mathscr{E}_3^{-1}(1 + 2r\mathscr{E}_1 B + \mathscr{E}_2).$$

This contradiction shows that

$$\operatorname{sign} s_{r+1}^*(A) = 1, \qquad \text{for all} \quad A > \hat{A}. \tag{2.77}$$

Fix some $A > \hat{A}$. The computation of $D(s^*) = D(s^*, A)$ produces the equations

$$0 = D_{s^*} = 2r \, \|H_{s^*}\|_{\mathbb{C}[0, \bar{t}_r^*]} - 2r \cdot B. \tag{2.78}$$

Finally,

$$\|H_{s^*, A}\|_{\mathbb{C}[0, \bar{t}_r^*]} = B. \tag{2.79}$$

In order to take the limit as $\varepsilon \to 0$, we need to show that the points $\{\bar{t}_i^* = \bar{t}_i^*(\varepsilon)\}_{i=0}^r$ remain uniformly separated:

$$|\bar{t}_i^*(\varepsilon) - \bar{t}_{i-1}^*(\varepsilon)| \geqslant \delta, \qquad i = 1, ..., r, \qquad \text{for all} \quad \varepsilon > 0. \tag{2.80}$$

Indeed, combining the estimate (2.45) with properties (2.57) and (2.79), we infer that for all $i = 0, ..., r$,

$$\begin{aligned} 2B &= |H_{s^*(\varepsilon)}(\bar{t}_i^*(\varepsilon)) - H_{s^*(\varepsilon)}(\bar{t}_{i-1}^*(\varepsilon))| \leqslant \|H_{s^*(\varepsilon)}\|_{\mathbb{C}[0, A]} \, |\bar{t}_i^*(\varepsilon) - \bar{t}_{i-1}^*(\varepsilon)| \\ &= \|W_{s^*(\varepsilon)}\|_{\mathbb{C}[0, A]} \, |\bar{t}_i^*(\varepsilon) - \bar{t}_{i-1}^*(\varepsilon)| \leqslant C \, |\bar{t}_i^*(\varepsilon) - \bar{t}_{i-1}^*(\varepsilon)|. \end{aligned} \tag{2.81}$$

Thus, we can put $\delta = 2B/C$ in (2.80).

The inequalities (2.45) for the norms $\{\|W_{s^*}^{(k)}\|_{\mathbb{C}[0,A]}\}_{k=0}^{r-1}$ and an application of the Arzela–Ascoli theorem enable us to choose a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_k\downarrow 0$, as $k\uparrow\infty$, such that

$$\begin{aligned} &\lim_{k\to\infty} t_i^*(\varepsilon_k)=t_i, && i=0,...,r;\\ &\lim_{k\to\infty} W_{s^*(\varepsilon_k)}(t)=W(t) && \text{in}\quad \mathbb{C}^r[0,1]. \end{aligned} \tag{2.82}$$

Then, by the inequalities (2.18),

$$\lim_{k\to\infty}\tau_i(\varepsilon_k)=\bar{t}_i:=\min\{t_i,1\},\qquad \lim_{k\to\infty}T_i(\varepsilon_k)=t_i,\qquad i=0,...,r. \tag{2.83}$$

Also, by our observation (2.56), $T_i=\bar{t}_i=t_i$, $i=0,...,r-1$. Put

$$\begin{aligned} &U(t)=(-1)^{i+m}\,|W(t)|, && \bar{t}_{i-1}\leqslant t\leqslant \bar{t}_i,\quad 1\leqslant i\leqslant r+1;\\ &H(t)=\int_0^t U(y)\,dy, && 0\leqslant t\leqslant 1, \end{aligned} \tag{2.84}$$

so $U=\lim_{k\to\infty}U_{s^*(\varepsilon_k)}$ in $\mathbb{L}_1[0,1]$, and $H=\lim_{k\to\infty}H_{s^*(\varepsilon_k)}$ in $\mathbb{C}[0,1]$. The comparison of the properties (2.59), (C) and (2.31) combined with the limiting relations (2.83) leads us to the conclusion that for $t\in[\bar{t}_{i-1},\bar{t}_i]$, $i=1,...,r+1$,

$$\frac{d}{dt}H(t)=U(t)=(-1)^{i+m}\,|W(t)|=(-1)^{i+m}\,[(-1)^{i+m}\,W(t)]=W(t). \tag{2.85}$$

Therefore, $H\in W^rH^\omega[0,\bar{t}_r]$, and

$$H^{(k)}(t)=W^{(k-1)}(t),\qquad t\in[0,1],\quad k=1,...,r. \tag{2.86}$$

By (2.80), the points $\{\bar{t}_i\}_{i=0}^r$ are separated by the constant δ. Let the coefficients $\{\alpha_i\}_{i=0}^r$ be determined from the system of linear equations

$$\sum_{i=0}^r \alpha_i[\bar{t}_i]^k=(-1)^m\,\delta_{m,k},\qquad k=0,...,r, \tag{2.87}$$

and the kernel K be introduced by the formula

$$K(t)=-\frac{1}{(r-1)!}\sum_{i=0}^r\alpha_i(\bar{t}_i-t)_+^{r-1}. \tag{2.88}$$

It also follows from the property (2.80) and the definition (2.20) that

$$\lim_{k\to\infty} K_{s^*(\varepsilon_k)}(t) = K(t) \qquad \text{in} \quad \mathbb{C}[0,1]. \tag{2.89}$$

Therefore, we can apply Lemma 1.3 to the family of functions

$$f_{s^*}(t) = W_{s^*(\varepsilon_k)}(t) - W_{s^*(\varepsilon_k)}(0), \qquad 0 \leqslant t \leqslant \tau_r^*(\varepsilon_k), \qquad k \in \mathbb{N},$$

extremal in problems (2.21). Lemma 1.3 implies that

$$\sup_{h\in H_0^\omega[0,\bar t_r]} \int_0^{\bar t_r} h(t)\, K(t)\, dt = \int_0^{\bar t_r} [H^{(r)} - H^{(r)}(0)]\, K(t)\, dt. \tag{2.90}$$

From the properties (2.57) with $\chi = (-1)^m$ and (2.79) of the functions $H_{s^*(\varepsilon)}$ we infer that

$$H(\bar t_i) = (-1)^{i+m} \, \|H(\bar t_i)\|_{\mathbb{C}[0,\bar t_r]} = (-1)^{i+m}\, B. \tag{2.91}$$

The point t_r becomes the rth zero of the derivative $(d/dt)\, H(t) = W(t)$ on the interval $[0, 1)$, if $\bar t_r = t_r$, i.e., $t_r < 1$.

It remains to rename the extremal functions and the points:

$$\tau_i := \bar t_i, \quad i = 0, ..., r; \qquad Z_B(t) := H(t), \qquad K_B(t) = K(t), \quad t \in [0, \bar t_r]. \tag{2.92}$$

The extension of the function $Z_B^{(r)}(t)$ to the entire interval $[0, 1]$ can be given by the formula

$$Z_B^{(r)}(t) = Z_B^{(r)}(\tau_r) + (-1)^{r+m+1} [\omega(t) - \omega(\tau_r)], \qquad t \in [\tau_r, 1]. \tag{2.93}$$

By Corollary 1.2 and (2.93), the function $Z_B^{(r)}$ has the full modulus of continuity on $[0, 1]$:

$$\omega(Z_B^{(r)}; t) = \omega(t) = \begin{cases} \omega(Z_B^{(r)}; t), & 0 \leqslant t \leqslant \tau_r; \\ |Z^{(r)}(t) - Z^{(r)}(0)|. \end{cases} \tag{2.94}$$

The derivatives $\{Z_B^{(k)}(t)\}_{k=0}^{r-1}$ are extended to $[0, 1]$ by continuity.

The proof of Theorem 2.1 is completed. ∎

In conclusion, we remark that from the definition of the kernel $K(t)$ in (2.1), (2.2) for $\{\tau_i = \tau_i(B)\}_{i=0}^{r}$ it follows that the kernel $K(t) = K_m(t)$ depends on m, $0 < m \leqslant r$. Then, the Korneichuk formula (2.6) ($0 < m \leqslant r$) and (2.11) for the rth derivative $Z_B^{(r)}(t)$ imply that the family of Zolotarev ω-polynomials $\{Z_B = Z_{B,r,m,\omega}\}_{B>0}$ is dependent on m in the case of *nonlinear modulii of continuity* ω.

3. COROLLARIES OF THEOREM 2.1

Fix $r, m \in \mathbb{N} : 0 < m \leqslant r$.

Throughout this section, $\{\tau_i(B)\}_{i=0}^r$ is the set of alternance points of the function $Z_B = Z_{B,\omega,r,m}$ on the interval $[0, \tau_r(B)]$, and the kernel $K_B(x)$, $0 \leqslant x \leqslant 1$, is defined by (2.1), (2.2) for the specified collection $\{\tau_i = \tau_i(B)\}_{i=0}^r$.

3.1. *The Uniqueness Property of Zolotarev ω-Polynomials*

In the following corollary we prove the uniqueness of the solution of the problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^rH^\omega[0,1], \qquad \|f\|_{\mathbb{C}[0,\tau_r(B)]} \leqslant B. \tag{3.1}$$

COROLLARY 3.1. *Let $B > 0$. The function Z_B is a unique solution of the problem* (3.1).

Proof. By Theorem 2.1, Z_B is a solution of the problem (3.1). From the identities (2.4) for $0 < m < r$ and (2.9) for $m = r$ we deduce the following necessary and sufficient conditions for a function $\hat{f}$ to be extremal in the problem (3.1):

$$\begin{aligned} &\text{(i)} \quad \hat{f}(\tau_j(B)) = (-1)^{m+j} B, \qquad j = 0, ..., r; \\ &\text{(ii)} \quad \sup_{h \in H_0^\omega} \int_0^{\tau_r(B)} h(x)\, K_B(x)\, dx = \int_0^{\tau_r(B)} [\hat{f}^{(r)}(x) - \hat{f}^{(r)}(0)]\, K_B(x)\, dx, \end{aligned} \tag{3.2}$$

By the Korneichuk lemma, the extremal function in (3.2), (ii) is unique:

$$\hat{f}^{(r)}(x) - \hat{f}^{(r)}(0) = Z_B^{(r)}(x) - Z_B^{(r)}(0), \qquad 0 \leqslant x \leqslant \tau_r(B) \tag{3.3}$$

Therefore,

$$\hat{f}(x) = Z_B(x) + \sum_{i=0}^{r} c_i x^i, \qquad 0 \leqslant x \leqslant \tau_r(B), \quad c_i \in \mathbb{R}, \quad i = 0, ..., r. \tag{3.4}$$

However, by (3.2), (i), the difference $\hat{f}(x) - Z_B(x)$ vanishes at $r+1$ distinct points $\{\tau_i(B)\}_{i=0}^r$. Consequently, the coefficients of the polynomial $\sum_{i=0}^r c_i x^i = \hat{f}(x) - Z_B(x)$ are zeroes. ∎

3.2. *Zolotarev ω-Polynomials with r Alternance Points on* $[0, 1]$

Let us introduce the class

$$W^rH^\omega[B] := \{ f \in W^rH^\omega[0,1] \mid \|f\|_{\mathbb{C}[0,1]} \leqslant B \}. \tag{3.5}$$

In the following corollary we show that $\tau_r(B) = 1$ for all sufficiently large B's.

COROLLARY 3.2. *Let the class* $W^rH^\omega[B]$ *be defined in* (3.5). *There exists such a constant* $M = M_{\omega, r, m} > 0$ *that* $\tau_r(B) = 1$ *for all* $B > M$, *and*

$$\sup_{f \in W^rH^\omega[B]} f^{(m)}(0) = Z_B^{(m)}(0), \qquad \text{for all} \quad B > M. \tag{3.6}$$

Proof. Let us introduce the set $\Gamma = \Gamma(B, r, m, \omega) \in \mathbb{R}_+$:

$$\Gamma := \left\{ B > 0 \,\middle|\, \frac{d}{dt} Z_B(\tau_r(B)) = 0 \right\}. \tag{3.7}$$

By the assertion (iii) of Theorem 2.1, if $\tau_r(B) < 1$, then $(d/dt)\, Z_B(\tau_r(B)) = 0$, and $B \in \Gamma$. Therefore,

$$B \notin \Gamma \Leftrightarrow \tau_r(B) = 1 \qquad \text{and} \qquad \frac{d}{dt} Z_B(\tau_r(B)) \neq 0. \tag{3.8}$$

LEMMA 3.2.1. *Let the set* Γ *be introduced by* (3.7). *Then,*

$$\text{(A)} \quad \inf_{B \in \mathbb{R}_+ \setminus \Gamma} B > 0; \qquad \text{(B)} \quad \sup_{B \in \Gamma} B < +\infty. \tag{3.9}$$

Proof. Let us show that Γ is nonempty. Indeed, otherwise, $\tau_r(B) = 1$ for all $B > 0$. Then, by Corollary 1.2.1, all functions $Z_B^{(r)}$ have the full modulus of continuity on $[0, \tau_r(B)] = [0, 1]$: $\omega(Z_B^{(r)}; t) = \omega(t)$, $0 \leqslant t \leqslant 1$, and

$$(-1)^{r+m+1} [Z_B^{(r)}(1) - Z_B^{(r)}(0)] = \omega(1), \qquad \text{for all} \quad B > 0.$$

The Arzela–Ascoli theorem enables us to choose such a sequence $B_k \downarrow 0$, as $k \uparrow \infty$, that $\lim_{k \to \infty} Z_{B_k} = Z$ in $\mathbb{C}[0, 1]$. Then, the contradicting properties $\|Z\|_{\mathbb{C}[0,1]} = 0$ and $|Z^{(r)}(1) - Z^{(r)}(0)| = \omega(1)$ of the limiting function Z prove that our assumption was wrong. Thus, the set Γ is nonempty and $\sup_{B \in \Gamma} B > 0$.

On the other hand, if $B \in \Gamma$, then the derivative $(d/dt)\, Z_B(t)$ has r distinct zeroes $\{\tau_i(B)\}_{i=1}^r$ on $[0, \tau_r(B)]$. Thus, $Z_B^{(r)}$ has a zero on $[0, \tau_r(B)]$, and by Proposition 1.6,

$$B = \|Z_B\|_{\mathbb{C}[0, \tau_r(B)]} \leqslant [\tau_r(B)]^r \, \|Z_B^{(r)}\|_{\mathbb{C}[0, \tau_r(B)]} \leqslant \omega(1). \tag{3.10}$$

Let us define the constant $M = M(\omega, r, m)$:

$$M := \sup_{B \in \Gamma} \|Z_B\|_{\mathbb{C}[0,1]}. \tag{3.11}$$

By (3.8), $\tau_r(B) = 1$, $B \in \mathbb{R}_+ \setminus \Gamma$. In particular, $\tau_r(B) = 1$ for all $B > M$. Therefore,

$$Z_B(\tau_i(B)) = (-1)^{i+m} \|Z_B\|_{\mathbb{C}[0,1]} = (-1)^{i+m} B, \qquad \text{for all} \quad B > M, \tag{3.12}$$

implying the extremal property (3.6) of the function Z_B for $B > M$. ∎

We remark that by (3.8), functions Z_B have the extremal property (3.6) for all $B \in \mathbb{R}_+ \setminus \Gamma$.

The following result follows immediately from Corollaries 3.1 and 3.2 on the uniqueness of the solution of the problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^rH^\omega[B],$$

for all $B \geqslant M$.

COROLLARY 3.2.2. *Let $\{\tau_i(B)\}_{i=0}^r$ be the alternance points of the extremal functions Z_B. Then, the mapping $B \mapsto \{\tau_i(B)\}_{i=0}^r$ is continuous on the interval $[M, +\infty)$.*

Let us identify the set Γ, the constant M, and the function Z_M in the case of a linear modulus of continuity $\omega(t) = t$. Let $C_r(x)$ be the Chebyshev polynomial of degree $r+1$ defined in (0.2),

$$L_r := \|C_r\|_{\mathbb{C}[0,1]} = \frac{2^{-2r-1}}{(r+1)!}, \quad \left\{T_i = \frac{1}{2}\left(1 + \cos\frac{\pi i}{r+1}\right)\right\}_{i=0}^{r+1}$$

be the collection of alternance points of $C_r(x)$ on the interval $[0,1]$, and $K_r := L_r \cdot T_r^{-(r+1)}$. Then, $\Gamma = (0, K_r]$, $M = K_r$, and for $B \in \Gamma$, the polynomial Z_B is given by the formula (0.4). The following section describes a similar phenomenon in Hölder classes.

3.3. *Zolotarev ω-Polynomials in Hölder Classes*

Fix α, $0 < \alpha \leqslant 1$, and consider the Hölder classes $W^rH^\alpha[0,1] := W^rH^{\omega_\alpha}[0,1]$, where $\omega_\alpha(t) = t^\alpha$.

The extremal functions $\{Z_B = Z_{B,r,m,\omega_\alpha}\}_{B>0}$ have the following specific feature. For a fixed $B > 0$, let $\tau_i = \tau_i(B)$, $i = 0, ..., r$, and

$$\mathscr{Z}(t) = Z_B(t), \qquad K(t) = K_B(t), \qquad 0 \leqslant \tau_r(B). \tag{3.13}$$

For all $\beta > 0$, put $B[\beta] := \beta^{r+\alpha}B$, $\tau_i[\beta] = \beta\tau_i$, $i = 0, ..., r$, and

$$Y_\beta(t) = \beta^{r+\alpha}\mathscr{Z}(t/\beta), \qquad V_\beta(t) = \beta^{r-1-m}K(t/\beta), \qquad 0 \leqslant t \leqslant \tau_r[\beta]. \tag{3.14}$$

By the definitions (2.1), (2.2) of the kernel $K(t)$ and (3.14) of the kernel $V_\beta(t)$,

$$V_\beta(t) = \sum_{i=0}^{r} \alpha_i[\beta](\tau_i[\beta] - t)_+^{r-1}, \tag{3.15}$$

and the coefficients $\{\alpha_i[\beta]=\beta^{-m}\alpha_i\}_{i=0}^{r}$ satisfy the equations

$$\sum_{i=0}^{r} \alpha_i[\beta](\tau_i[\beta])^j=\delta_{m,j}, \qquad j=0, ..., r. \tag{3.16}$$

Therefore, for any function $f\in W^rH^\alpha[0, \tau_r[\beta]]$, the familiar identity holds:

$$f^{(m)}(0)=\sum_{i=0}^{r} \alpha_i f(\tau_i[\beta])+\int_0^{\tau_r[\beta]} f^{(r)}(t)V_\beta(t)\,dt. \tag{3.17}$$

By Corollary 1.2.2, applied to the dylation V_β of the kernel K, and the assertion (i) of Theorem 2.1,

$$\sup_{h\in H_0^\omega[0,\tau_r[\beta]]} \int_0^{\tau_r[\beta]} h(t)\,V_\beta(t)\,dt=\int_0^{\tau_r[\beta]} (Y_\beta(t)-Y_\beta(0))\,V_\beta(t)\,dt. \tag{3.18}$$

By the assertion (ii) of Theorem 2.1 and the definition (3.14) of the function Y_β,

$$Y_\beta(\tau_i[\beta])=(-1)^{i+m}\,\|Y_\beta\|_{\mathbb{C}[0,\tau_r[\beta]]}=(-1)^{i+m}\,B[\beta], \qquad i=0, ..., r. \tag{3.19}$$

The properties (3.18) and (3.19) along with the identity (3.17) imply that the function $Y_\beta(t)$ has the extremal property

$$Y_\beta^{(m)}(0)=\sup\{f^{(m)}(0)\,|\,f\in W^rH^\alpha[0,\tau_r[\beta]],\ \|f\|_{\mathbb{C}[0,\tau_r[\beta]]}\leqslant B[\beta]\}. \tag{3.20}$$

This property will be used in the proof of the following results.

LEMMA 3.3. *Let the set $\Gamma=\Gamma(r, m, \omega_\alpha)$ be introduced in* (3.7) *and $M=M_{r,m,\omega_\alpha}$ be the constant defined in* (3.11) *for $\omega(t)=\omega_\alpha(t)$. Let $\{\tau_i(B)=\tau_i(B, r, m, \alpha)\}_{i=0}^{r}$ be the points of alternance of the function $Z_B=Z_{B,r,m,\omega_\alpha}$. Then,*

$$M\in\Gamma \qquad \textit{and} \qquad \tau_r(M)=1.$$

Proof. For each $B\in\Gamma$, let us introduce the function $X_B(t)$ and the kernel $W_B(t)$:

$$\begin{aligned} X_B(t)&=[\tau_r(B)]^{-r-\alpha}\,Z_B(\tau_r(B)t), \qquad 0\leqslant t\leqslant 1;\\ W_B(t)&=[\tau_r(B)]^{m+1-r}\,K_B(\tau_r(B)t), \qquad 0\leqslant t\leqslant 1. \end{aligned} \tag{3.21}$$

Note that

$$\frac{d}{dt}X_B(1)=[\tau_r(B)]^{1-r-\alpha}\frac{d}{dt}Z_B(\tau_r(B))=0, \qquad \text{for all} \quad B\in\Gamma. \tag{3.22}$$

Let $A_B=\|X_B\|_{\mathbb{C}[0,1]}=[\tau_r(B)]^{-r-\alpha}B$. As we explained in (3.18)–(3.20), the function X_B is extremal in the problem

$$f^{(m)}(0)\to\sup, \qquad f\in W^rH^\alpha[A_B]. \tag{3.23}$$

First, let us show that the inclusion $M\in\Gamma$ implies the property $\tau_r(M)=1$. Indeed, if $\tau_r(M)<1$, then $A_M>M$ and, consequently, $A_M\notin\Gamma$. Therefore, by Corollaries 3.1 and 3.2, $\tau_r(A_M)=1$ and the function Z_{A_M} is a unique solution of the problem (3.23) for $B=M$. Thus, $X_M=Z_{A_M}$. Then, by the property (3.22), $(d/dt)\,Z_{A_M}(1)=(d/dt)\,X_M(1)=0$, and the definition (3.7) of the set Γ implies $A_M\in\Gamma$. Then, the contradicting inclusions $A_M\in\Gamma$ and $A_M\notin\Gamma$ prove that $\tau_r(M)=1$.

It remains to eliminate the case $M\notin\Gamma$. In this case, let us consider such a subsequence $\{B_i\}_{i\in\mathbb{N}}$ that $B_i\in\Gamma$, $i\in\mathbb{N}$, $\lim_{i\to\infty}B_i=M$, and

$$\begin{aligned}
&\lim_{i\to\infty}(\tau_0(B_i),\ldots,\tau_r(B_i))=(T_0,\ldots,T_r);\\
&\lim_{i\to\infty}X_{B_i}=\mathscr{X}(t) \qquad \text{in} \quad \mathbb{C}^r[0,1];\\
&\lim_{i\to\infty}W_{B_i}(t)=W(t)=\sum_{i=0}^{r}\alpha_i(T_i-t)_+^{r-1} \qquad \text{in} \quad \mathbb{C}^{r-1}[0,1],
\end{aligned} \tag{3.24}$$

where the coefficients $\{\alpha_i\}_{i=0}^r$ satisfy Eq. (2.1) for $\{\tau_i=T_i\}_{i=0}^r$.

Let $D=MT_r^{-r-\alpha}$. By Lemma 1.3, applied to the family of kernels $\{W_{B_i}\}_{i\in\mathbb{N}}$, the function $\mathscr{X}(t)$ inherits the properties of functions $\{X_{B_i}\}_{i\in\mathbb{N}}$:

$$\begin{aligned}
&\text{(i)} \quad \mathscr{X}(T_i)=(-1)^{i+m}\|\mathscr{X}\|_{\mathbb{C}[0,1]}=(-1)^{i+m}D, \qquad i=0,\ldots,r;\\
&\text{(ii)} \quad \sup_{h\in H_0^\omega[0,1]}\int_0^1 h(t)\,W(t)\,dt=\int_0^1[X^{(r)}(t)-X^{(r)}(0)]\,W(t)\,dt;\\
&\text{(iii)} \quad \frac{d}{dt}\mathscr{X}(1)=0.
\end{aligned} \tag{3.25}$$

Therefore, the function $\mathscr{X}(t)$ is extremal in the problem

$$f^{(m)}(0)\to\sup, \qquad f\in W^rH^\alpha[D]. \tag{3.26}$$

Since $D\geqslant M$, our assumption $M\notin\Gamma$ implies that $D\notin\Gamma$, as well. Therefore, by Corollary 3.2, the function Z_D is a unique solution of the problem

(3.26). Thus, $X(t)=Z_D(t)$. Then, the property (3.25), (iii) implies that $D\in\Gamma$. Once again, the contradicting inclusions $D\in\Gamma$ and $D\notin\Gamma$ lead us to the conclusion that our assumption $M\notin\Gamma$ was wrong.

3.4. *Chebyshev ω-Polynomials in $W^rH^\alpha[0,1]$*

In the following corollary we construct the analog $C(x)=C_{r,m,\alpha}(x)$ of the Chebyshev polynomial in the Hölder space $W^rH^\alpha[0,1]$. Like in the linear case $\omega(t)=At$, we describe all extremal functions in the problem

$$f^{(m)}(0)\to\sup,\qquad f\in W^rH^\omega[B] \tag{3.27}$$

for $B\geqslant L$.

COROLLARY 3.4. *Let* $0<\alpha\leqslant 1$.

1. *There exist a constant* $L=L_{\alpha,r,m}>0$, *the collection of points* $\{T_i=T_i(r,m,\alpha)\}_{i=0}^{r+1}: 0=T_0<T_1<\cdots<T_r<T_{r+1}=1$, *and the function* $C(x)=C_{\alpha,r,m}(x)$ *endowed with the properties*

(i) $C(T_i)=(-1)^{i+m}\,\|C\|_{\mathbb{C}[0,1]}=(-1)^{i+m}L\qquad i=0,...,r+1;$

(ii) $$\sup_{h\in H_0^\alpha[0,T_r]}\int_0^{T_r}h^{(r)}(t)\,\mathcal{K}(t)\,dt=\int_0^{T_r}[C^{(r)}(t)-C^{(r)}(0)]\,\mathcal{K}(t)\,dt, \tag{3.28}$$
where $\mathcal{K}(t)$ *is defined by* (2.1), (2.2) *for* $\{\tau_i=T_i\}_{i=0}^{r}$;

(iii) $\omega(C^{(r)};t)=\omega_\alpha(t),\qquad 0\leqslant t\leqslant 1.$

2. *For any* $B\geqslant L$ *there exists a collection of points* $\{\tau_i=\tau_i(r,m,\alpha,B)\}_{i=0}^{r}$: $0=\tau_0<\tau_1<\cdots<\tau_r\leqslant 1$, *and a function* $Z_B(t)\in W^rH^\omega[0,1]$ *with the properties*

(i) $Z_B(\tau_i)=(-1)^{i+m}\,\|Z_B\|_{\mathbb{C}[0,1]}=(-1)^{i+m}B,\qquad i=0,...,r;$

(ii) $$\sup_{h\in H_0^\alpha[0,\tau_r]}\int_0^{\tau_r}h^{(r)}(t)\,K_B(t)\,dt=\int_0^{\tau_r}[Z_B^{(r)}(t)-Z_B^{(r)}(0)]\,K_B(t)\,dt, \tag{3.29}$$
where $K_B(t)$ *is defined by* (2.1), (2.2);

(iii) $\omega(Z_B^{(r)};t)=\omega_\alpha(t),\qquad 0\leqslant t\leqslant 1.$

3. *For any* $B\geqslant L$, *the function* $Z_B(x)=Z_{\alpha,r,m,B}(x)$ *is extremal in* (3.27).

Proof. Let $M = M(r, m, \omega_\alpha)$ be the constant defined in (3.11) for $\omega = \omega_\alpha$. By Lemma 3.3, $\tau_r(M) = 1$, so the function Z_M is defined on the whole interval $[0, 1]$. Let

$$\mathscr{Z}^{(r)}(t) = \begin{cases} Z_M^{(r)}(t), & 0 \leqslant t \leqslant 1; \\ Z_M^{(r)}(0) + (-1)^{r+m+1}\,\omega(t), & t > 1. \end{cases} \tag{3.30}$$

Then, the extension $\mathscr{Z}(t)$ of the function $Z_M(t)$ from the interval $[0, 1]$ to the entire half-line $\mathbb{R}_+$ is given by the formula

$$\mathscr{Z}(t) = \sum_{i=0}^{r-1} \frac{Z_M^{(i)}(0)}{i!}\, t^i + \frac{1}{r!} \int_0^t \mathscr{Z}^{(r)}(x)(t-x)_+^{r-1}\, dx, \qquad t \in \mathbb{R}_+. \tag{3.31}$$

Let us show that $\mathscr{Z} \in W^rH^\alpha(\mathbb{R}_+)$. By the definition, we have the inclusions $\mathscr{Z}^{(r)}|_{[0,1]} \in H^\alpha[0, 1]$ and $\mathscr{Z}^{(r)}|_{[1,+\infty)} \in H^\alpha[1, +\infty)$. It remains to verify the inequality

$$|\mathscr{Z}^{(r)}(t_2) - \mathscr{Z}^{(r)}(t_1)| \leqslant \omega_\alpha(t_2 - t_1), \qquad \text{for} \quad t_1 \in [0, 1], \qquad t_2 > 1. \tag{3.32}$$

In this case, the definition (3.30) of $\mathscr{Z}^{(r)}$ and the concavity of $\omega_\alpha(t)$ lead us to the following chain of inequalities:

$$\begin{aligned} |\mathscr{Z}^{(r)}(t_2) - \mathscr{Z}^{(r)}(t_1)| &= |\mathscr{Z}^{(r)}(t_2) - \mathscr{Z}^{(r)}(1)| + |\mathscr{Z}^{(r)}(1) - \mathscr{Z}^{(r)}(t_1)| \\ &\leqslant [\omega_\alpha(t_2) - \omega_\alpha(1)] + \omega_\alpha(1 - t_1) \\ &\leqslant [\omega_\alpha(t_2 - t_1) - \omega_\alpha(1 - t_1)] \\ &\quad + \omega_\alpha(1 - t_1) = \omega_\alpha(t_2 - t_1) \end{aligned} \tag{3.33}$$

Moreover, by the definition (3.30) and Corollary 1.2.1 of the Korneichuk lemma, the function $\mathscr{Z}^{(r)}(t)$ has the full modulus of continuity on $\mathbb{R}_+$:

$$\omega(\mathscr{Z}^{(r)}; t) = \omega(t) = \begin{cases} \omega(Z_M^{(r)}; t), & 0 \leqslant t \leqslant 1; \\ |Z^{(r)}(t) - Z^{(r)}(0)|, & t > 1. \end{cases} \tag{3.34}$$

For each $\beta \in [0, 1]$ let us introduce the function $P_\beta \in W^rH^\omega(\mathbb{R}_+)$:

$$P_\beta(t) := \beta^{r+\alpha} \mathscr{Z}(t/\beta), \qquad t \in \mathbb{R}_+. \tag{3.35}$$

Put

$$M[\beta] := \beta^{r+\alpha} M, \qquad \tau_i[\beta] := \beta \tau_i(M), \tag{3.36}$$

where $M=\|Z_M\|_{\mathbb{C}[0,1]}$ and $\{\tau_i(M)\}_{i=0}^r$ are the points of alternance of Z_M. The points $\{\tau_i[\beta]\}_{i=0}^r$ are the alternance points of P_β on the interval $[0,\beta]$:

$$P_\beta(\tau_i[\beta])=(-1)^{i+m}\|P_\beta\|_{\mathbb{C}[0,\beta]}=(-1)^{i+m}M[\beta]. \qquad (3.37)$$

By our observation (3.20), the restriction $P_\beta|_{[0,\beta]}$ has the extremal property

$$P_\beta^{(m)}(0)=\sup\{f^{(m)}(0)\,|\,f\in W^rH^\alpha[0,\beta],\ \|f\|_{\mathbb{C}[0,\beta]}\leqslant M[\beta]\}. \qquad (3.38)$$

Also note that by Corollary 3.3, the derivative $(d/dt)\,P_\beta(t)$ has the rth zero at the point $\tau_r[\beta]=\beta$:

$$\frac{d}{dt}P_\beta(\beta):=\beta^{r+\alpha-1}\frac{d}{dt}Z_M(1)=0, \qquad \text{since} \quad M\in\Gamma.$$

From the monotonicity of $P_\beta^{(r)}(t)$ on $\mathbb{R}_+$ and Rolle's theorem it follows that the derivative $(d/dt)\,P_\beta(t)$ vanishes only at r points $\{\tau_i[\beta]\}_{i=1}^r$ and

$$\operatorname{sign}\frac{d}{dt}P_\beta(t)=(-1)^{i+m+1}, \qquad t\in(\tau_i[\beta],\tau_{i+1}[\beta]), \qquad i=0,...,r, \quad (3.39)$$

where $\tau_{r+1}[\beta]:=\infty$. In particular, the function $(-1)^{r+m}P_\beta(t)$ strictly decreases from $\|P_\beta\|_{\mathbb{C}[0,\beta]}$ to $-\infty$, as t increases from $\tau_r[\beta]$ to $+\infty$.

Let us introduce the parameter $k(\beta)$ by the equation

$$P_\beta(1)=k(\beta)\,P_\beta(\tau_r[\beta]). \qquad (3.40)$$

By the definition (3.35) of the function P_β and the property (3.37),

$$\begin{aligned} k(\beta)&=P_\beta(1)[P_\beta(\tau_r[\beta])]^{-1}\\ &=\beta^{r+\alpha}Z_M(1/\beta)[(-1)^{r+m}\beta^{r+\alpha}M]^{-1}\\ &=(-1)^{r+m}Z_M(1/\beta)\,M^{-1}. \end{aligned} \qquad (3.41)$$

This expression for $k(\beta)$ coupled with the property (3.39) for $i=r$ implies that the function $k(\beta)$ is continuous and strictly decreases from 1 to $-\infty$, as β decreases from 1 to 0. In particular, there exists such a $\beta^*\in(0,1)$ that

$$k(\beta^*)=-1, \qquad \text{and} \qquad -1\leqslant k(\beta)\leqslant 1, \qquad \text{if} \quad \beta^*\leqslant\beta\leqslant 1. \qquad (3.42)$$

Therefore, by (3.39) for $i=r$ and (3.42), the monotonicity of $P_\beta(t)$ on the interval $[\tau_r[\beta],1]$ implies that for all $\beta\in[\beta^*,1]$, and $t\in[\tau_r(B),1]$

$$|P_\beta(t)|\leqslant\max\{|P_\beta(\tau_r[\beta]),|k(\beta)|\cdot|P_\beta(\tau_r[\beta])|\}=\|P_\beta\|_{\mathbb{C}[0,\beta]}. \qquad (3.43)$$

Thus, we have the following refinements of the property (3.37),

$$P_\beta(\tau_i[\beta]) = (-1)^{i+m} \|P_\beta\|_{\mathbb{C}[0,1]} = (-1)^{i+m} M[\beta], \qquad i = 0, ..., r, \quad (3.44)$$

and the property (3.38),

$$\sup_{f \in W^r H^\alpha[M[\beta]]} f^{(m)}(0) = P_\beta^{(m)}(0). \quad (3.44)$$

Finally, by (3.37) and (3.42), $P_{\beta^*}(1) = k(\beta^*)\, P_\beta(\tau_r[\beta]) = (-1)^{r+m+1} \|P_{\beta^*}\|_{\mathbb{C}[0,1]}$, so the function $P_{\beta^*}(t)$ has precisely $r+2$ points of alternance $\{\tau_i[\beta^*]\}_{i=0}^{r}$ and $\tau_{r+1}[\beta^*] = 1$. Put

$$C(x) := P_{\beta^*}(x), \qquad x \in [0,1], \quad L := \|C\|_{\mathbb{C}[0,1]}. \quad (3.45)$$

Summarizing, the family of functions P_β, $\beta^* \leqslant \beta \leqslant 1$, constitutes the set of solutions of the problem (3.27) for $B \in [L, M] = [\|C\|_{\mathbb{C}[0,1]}, \|Z_M\|_{\mathbb{C}[0,1]}]$. By Corollary 3.2, the functions Z_B, $B > M$, are extremal in the problem (3.27) for $B > M$.

3.5. *Full Solution of the Kolmogorov–Landau Problem in $W^2H^\omega[0,1]$*

Before characterizing extremal functions in the problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^2 H^\omega[B] \quad (3.46)$$

for $m = 1, 2$, and all $B > 0$, we make the following observation on the possibility of functional extensions from the class $W^2H^\omega[0,1]$ to the class $W^2H^\omega(\mathbb{R}_+)$ without increasing the $\mathbb{L}_\infty$-norm.

Suppose that the derivative $(d/dt)\, g(t)$ of a function $g \in W^2H^\omega[0,1]$ has two zeroes t_1, t_2, $0 \leqslant t_1 < t_2 \leqslant 1$. Let $\Delta = t_2 - t_1$. Then, the extension $E(g; t_1, t_2; \cdot)$ of the function $g(\cdot)$ from the interval $[0, t_2]$ to the entire half-line $\mathbb{R}_+$ is given by the formula

$$E(g; t_1, t_2; t) = \begin{cases} g(t), & 0 \leqslant t \leqslant t_2; \\ g(t - 2n\,\Delta), & t_2 + (2n-1)\,\Delta \leqslant t \leqslant t_2 + 2n\,\Delta, \quad n \in \mathbb{N}; \\ g(2t_2 + 2(n+1)\,\Delta - t), & t_2 + 2(n-1)\,\Delta \leqslant t \leqslant t_2 + (2n-1)\,\Delta. \end{cases} \quad (3.47)$$

The properties $(d/dt)\, g(t_1) = (d/dt)\, g(t_2) = 0$ assure the continuity of $(d/dt)\, g(t)$ on $\mathbb{R}_+$. In addition,

$$\omega(E^{(2)}(g; t_1, t_2; \cdot); t) = \begin{cases} \omega(g; t), & 0 \leqslant t \leqslant t_2; \\ \omega(g; t_2), & t > t_2. \end{cases} \quad (3.48)$$

Thus, $E(g; t_1, t_2; t) \in W^2H^\omega(\mathbb{R}_+)$. Also notice that

$$\|E(g; t_1, t_2; \cdot)\|_{\mathbb{L}_\infty(\mathbb{R}_+)} = \|g\|_{\mathbb{C}[0, t_2]}. \tag{3.49}$$

Therefore, we extended the function g to the entire half-line $\mathbb{R}_+$ without leaving the class $W^2H^\omega(\mathbb{R}_+)$ and increasing the L_∞ norm.

Fix $m, m=1, 2$, and $B>0$. In Theorem 2.1 for $r=2$ we proved the existence of such a function Z_B with three points of alternance $\{\tau_i(B)\}_{i=0}^2$ that Z_B is extremal in the problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^2H^\omega[0, 1], \qquad \|f\|_{\mathbb{C}[0, \tau_2(B)]} \leqslant B. \tag{3.50}$$

Therefore, if $\tau_2(B)=1$, then the function Z_B is extremal in the problem (3.46).

By the assertion (iii) of Theorem 2.1, $(d/dt)\, Z_B(\tau_2(B))=0$, if $\tau_2(B)<1$. Besides, the derivative $(d/dt)\, Z_B(t)$ vanishes at the interior point $\tau_1(B)$ of extremum of $Z_B(t)$. Therefore, by (3.51), the restriction $E(Z_B; \tau_1(B), \tau_2(B); t)|_{[0,1]}$ is extremal in problem (3.47), if $\tau_2(B)<1$.

Kolmogorov–Landau problems in functional classes $W^2H^\omega(\mathbb{R})$ and $W^2H^\omega(\mathbb{R}_+)$ are solved in [4].

4. CONCLUDING REMARKS

The complete solution of the Kolmogorov–Landau problem

$$f^{(m)}(0) \to \sup, \qquad f \in W_\infty^{r+1}[0, 1], \qquad \|f\|_{\mathbb{C}[0,1]} \leqslant B,$$

in the Sobolev class $W_\infty^{r+1}[0, 1]$ was given by S. Karlin [7] who constructed the family of extremal *Zolotarev perfect splines* $\{\mathscr{Z}_B\}_{B>0}$. For each $B>0$, the function $\mathscr{Z}_B$ of the norm B has $n=n(B)\geqslant 0$ knots and oscillates $n+r+1$ times between $B=\|\mathscr{Z}_B\|_{\mathbb{C}[0,1]}$ and $-B$. It can be seen from the corresponding numerical differentiation formulae (see [8]) that the complete solution of the Kolmogorov–Landau problem in $W^rH^\omega[0, 1]$, requires an appropriate generalization of the notion of *perfect splines* in functional classes $W^rH^\omega[0, 1]$.

In our paper [2] we give the characterization of the structure and the description of various properties of extremal functions in the problem

$$\int_a^b h(t)\, \psi(t)\, dt \to \sup, \qquad h \in H_0^\omega[a, b], \tag{$*$}$$

for kernels $\psi \in \mathbb{L}_1[a,b]$ with a zero mean on $[0,1]$ and a finite or ordered countable set of points of sign change on the interval $[a,b]$, $-\infty \leqslant a < b \leqslant +\infty$. The extremal functions of the problem $(*)$ feature as the rth derivatives of solutions of the problem

$$f^{(m)}(0) \to \sup, \qquad f \in W^rH^\omega(I), \qquad \|f\|_{\mathbb{L}_\infty[I]} \leqslant B, \tag{P.1}$$

for $0<m<r$ and $I=[0,1]$, $\mathbb{R}$, $\mathbb{R}_+$. The problem (P.1) for $m=r$ and $I=[0,1]$ or $I=\mathbb{R}_+$ necessitated our solution in [3] of the problem $(*)$ for kernels with nonzero means.

The solution and corresponding numerical differentiation formulae in the pointwise Kolmogorov–Landau problem

$$f^{(m)}(\xi) \to \sup, \qquad f \in W_\infty^{r+1}[0,1], \qquad \|f\|_{\mathbb{C}[0,1]} \leqslant B, \tag{P.2}$$

were found by A. Pinkus [11]. In [3] we also describe the extremal functions of the problem

$$\int_a^0 h(t)\,\psi_1(t)\,dt + \int_0^b h(t)\,\psi_2(t)\,dt \to \sup, \qquad h \in H^\omega[a,b], \quad h(0)=E \in \mathbb{R}, \tag{$**$}$$

for $a<0<b$, and integrable kernels ψ_1 and ψ_2 with finite or ordered countable set of points of sign change on $[a,0]$ and $[0,b]$, respectively. As an example of an application of the extremal functions of the problem $(**)$, we mention the version of the problem (P.2) of maximizing the rth derivative of functions from $W^rH^\omega[a,b]$ at an interior point $\xi \in (a,b)$. Since extremal functions of problems $(*)$ and $(**)$ generalize standard perfect polynomial splines; we call them *the perfect ω-splines.*

Finally, the formulations of some results and referrences to our papers in the area of Kolmogorov–Landau inequalities in functional classes $W^rH^\omega[I]$ may be found in [3] and [4].

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